This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Phys
227
8

Cross. Course in Elementary Physics. 1873

Phys 227.8

HARVARD COLLEGE

VE RI TAS

SCIENCE CENTER
LIBRARY

Wm Rotch Ware

1/3

COURSE

IN

ELEMENTARY PHYSICS.

BY

CHARLES R. CROSS,

Assistant Professor of Physics in the Massachusetts Institute of Technology.

BOSTON:
PRESS OF A. A. KINGMAN.
1873.

COURSE

IN

ELEMENTARY PHYSICS.

BY

CHARLES R. CROSS,

Assistant Professor of Physics in the Massachusetts Institute of Technology.

BOSTON:
PRESS OF A. A. KINGMAN.
1873.

Phys 227.8

HARVARD COLLEGE
Sept. 3, 1937
LIBRARY

From the Library of
William Rotch Ware

Entered according to Act of Congress, in the year 1878, by
CHARLES R. CROSS,
in the office of the Librarian of Congress, at Washington.

PREFACE.

The present pamphlet comprises the text of the first few chapters of an elementary work on Physics, designed to supplement the lectures delivered to the students of the Massachusetts Institute of Technology during the first two years of their course. It is intended to impart such a general knowledge of the science as will enable them with advantage to continue the pursuit of the subject in the more advanced work of the Physical Laboratory.

It will be the aim of the author to present the student with a concise statement of the leading principles of the science, and to explain the methods of investigation by which they have been ascertained.

In the preparation of the work constant reference has been made to the original memoirs as well as to the standard text books upon Physics. In the chapters relating to mechanics, the *Elements of Mechanical Philosophy*, by Prof. W. B. Rogers, long out of print, has been of the greatest service, and valuable hints have been derived from the elementary works of Smith, Peck and Mayer. The text-books of Thomson and Tait, Rankine, Todhunter and Price have furnished many suggestions.

A list of standard text-books upon General Physics will be added hereafter, as well as a collection of physical tables. The references appended to various chapters are not intended to be exhaustive, but may serve as aids to the student who is desirous of investigating the subject more thoroughly.

The work will be put in a more permanent form, and properly illustrated, as soon it reaches a suitable stage of progress. The present sheets have been printed for the temporary use of the students.

Boston, *January*, 1873.

ELEMENTARY PHYSICS.

CHAPTER I.

GENERAL METHODS OF PHYSICAL SCIENCE.

1. Physical Sciences. All those branches of human knowledge which have for their end the study of material objects and the phenomena of the external world, are included among the Physical Sciences. They may be divided into two general classes; first, those which aim simply at a classification of objects, and, secondly, those which endeavor to ascertain the laws of the phenomena of the material universe. The former class comprehends the various branches of Natural History. It includes Zoology, which describes and classifies the various forms of animal life; Systematic Botany, dealing with the classification of plants; Anatomy, both vegetable and animal, describing the structure of the organs of living things; and Mineralogy, which concerns itself with the constitution of the rock masses of the globe. The second class comprises Mechanical Philosophy or Physics proper (of which Astronomy is a branch) and Chemistry, which investigate the general laws of natural actions; Dynamical Geology, which studies the laws regulating the structure of the earth, and Physiology, whose province is the determination of the functions and mode of action of the organs of animals and plants.

2. Physics. In its original and most extended sense, Physics, or Natural Philosophy, as it is sometimes called, comprehended the entire range of physical as distinguished from mental science. The term is still occasionally used in a comprehensive manner to denote the second class of sciences of which we have spoken.

In its limited sense, and the one in which it is used in this work, it is applied only to that branch of science which investigates the laws of physical actions in which there is no permanent change in the general properties of the body acted upon. It is thus distinguished from Chemistry, which deals with actions in which the properties of the body affected are permanently modified or entirely changed. A few illustrations will suffice to make this plain. An unsupported body falls to the earth. Here we have a physical action in the motion produced, but the general properties of the body are not altered thereby, for it has the same weight, color and other characteristics as before. The law of its downward motion is then a subject lying within the province of physics. If the body is elastic it rebounds on reaching the earth; there is still no change of its general properties, so that the laws of elasticity are physical and not chemical. The motion ensuing when a piece of iron is brought near to a magnet, and the vibrations of a musical string, are additional examples. On the other hand, if a spark be applied to a mixture of oxygen and hydrogen they unite with an explosion, forming an entirely new substance, water, possessing properties totally different from those of either of the gases composing it. This, then, is a chemical change. Again, iron when exposed to moist air rusts, because of the union of oxygen with the metal; a new body, oxide of iron, is thus obtained, unlike either constituent. The laws of such changes are chemical laws. As would naturally be expected, there are many cases in which it is difficult to decide whether a given action is chemical or physical, or a mixture of both. We also have occasion to examine the physical effects of chemical changes, so that there are lines of investigation in which it is difficult to decide where one ends and the other begins.

3. Deductive and Inductive Sciences. The various sciences may be divided into two general classes, according to the mode of reasoning employed in the investigation of the facts pertaining to them. These methods are known as *deductive* and *inductive reasoning*, and the sciences characterized by their respective use as the Deductive and the Inductive Sciences. Deductive reasoning starts with certain general fundamental facts and definitions, and from these evolves truths which follow necessarily from the premises. Induction observes particular facts, and by analyzing and comparing them rises to more general truths. Deduction reasons from the general to the particular; induction from the particular to the general. Pure Mathematics furnishes us with an example of a deductive science. Certain definitions are laid down, and from these in connection with the axioms, or fundamental truths relating to magnitude, the whole science is evolved without the necessity of going outside of the mind itself for any portion of the demonstration. The most abstruse theorems follow directly from the fundamental simple ones. The Physical Sciences, on the contrary, are based on induction. Their end is to form a natural classification of objects, and to ascertain the general laws of the forces acting upon matter, and in no way can this be done except by the comparison of numerous particular cases. Hence we must reason from these to general laws. For example, a natural classification of animals can be attained only by observing the different species, noting their fundamental points of agreement and difference, and thus arranging them in groups connected with each other by certain structural peculiarities, which at the same time separate them from all other animals.

Having thus briefly stated the difference between these two kinds of reasoning, let us consider more fully the process of investigation by which the general laws of the physical world are ascertained and logically arranged.

4. Physical Law, Theory, Hypothesis. We shall frequently have occasion to use the terms *law*, *theory* and *hypothesis*. By a physical law is meant simply the constant method according to which a cause acts. "It is the custom of philosophers whenever they can trace regularity of any kind to call the general proposition which expresses the nature of that regularity a law, as when in mathematics we speak of the law of decrease of the successive terms of a converging series."[1] A theory is a true, complete and philosophical explanation of connected phenomena and their laws; as, for example, the theory of gravitation or the theory of sound. A hypothesis is a supposition as to the nature of any law or cause made in the absence of absolute proof, as, for example, the hypothesis of an electric fluid. When a hypothesis is extended to include a number of phenomena, and is proved to be true, it becomes a theory. The term theory is frequently applied more loosely than in the above definition to denote any plausible and clearly defined hypothesis.

5. Observation and Experiment. The facts which form the basis of physical science are ascertained in two ways, by *observation* and by *experiment*. Observation is the careful examination of phenomena as they occur in the order of nature; experiment is the production of phenomena by causes under our own control. It is evident that the latter method of investigation is a great extension of the former, and will often enable us to ascertain laws which could never be known without its aid. To illustrate by an example, suppose that we wish to know which of the components of the atmosphere, oxygen or nitrogen, enables it to support animal life. To ascertain this two processes are open. We may (1) notice which constituent of the atmosphere disappears when an animal is immersed in a limited quantity of air, or (2) we may immerse animals in oxygen and in nitrogen and compare the effects. Both of these methods require us to perform an experiment, and as nature does not furnish either oxygen or nitrogen in a pure state, and also as the oxygen consumed by animals from the atmosphere is replaced by that given out by plants, we could never have ascertained the truth in question from pure observation. Indeed, without the process of experimentation we could not even know of the existence of two ingredients in air. In some sciences, however, as Astronomy, the former method is the only one possible.

---

[1] This quotation, together with several others in the present chapter, is from Mill's *Inductive Logic*.

6. Induction. Having by these methods examined a large number of phenomena, we next approach the process by which we determine the causes of these phenomena and their laws of action. The great logical principles upon which our procedure is based are (1) that "every fact which has a beginning has a cause," and (2) that "the course of nature is uniform," the first of these being an intuitive idea, the second a truth ascertained from universal experience. Hence, when under certain influencing circumstances a given event occurs, we are entitled to infer that it will recur under the same circumstances, and it is the first object of science to ascertain exactly what these circumstances are, for they are the *causes* of the phenomena. In physical science we limit ourselves strictly to what are known as *physical causes*, that is the phenomena which are invariably and necessarily followed by some other phenomenon. The criterion by which we know that any combination of phenomena is the cause of another phenomenon is, that between the first and last there is invariable and *unconditional* sequence. It is not sufficient that the sequence be invariable, for in this case both events might be consequents of a single cause. Thus day and night invariably follow each other, but do not stand in the relation of cause and effect. The antecedence of the supposed cause must also be unconditioned, that is, its relation to the effect must be demonstrably such that without it the effect could not be produced. This demonstration can be given only by experiment or by the process of deduction. The constant endeavor of science is to find the smallest number of causes that will account for all natural phenomena, or as Mill expresses it, to find "the fewest and simplest assumptions which, being taken for granted, the whole existing order of nature would result." These primary assumptions we call Laws of Nature.

The causes of the phenomena studied by the processes of observation and experiment are to be ascertained only by a rigorous analysis and comparison of the circumstances attending their production. By investigating a number of isolated cases and ascertaining those circumstances which are present in all, we obtain a more general result, which we may infer to be true in analogous instances, even though these may be beyond the range of our present observation.

A single example will illustrate this. Experimental tests show that when oxygen and hydrogen unite to form water, the proportions are invariably 8 parts by weight of oxygen to 1 of hydrogen. So also when chlorine and hydrogen unite to form chlorhydric acid, the invariable proportion of the constituents is 35.5 parts of the former to 1 of the latter. In all other chemical compounds which we have examined, a like constancy of proportion is observed. Hence we lay it down as a general law, that the proportions in which substnaces unite to form a chemical compound are fixed, definite and invariable; and are justified in predicting that the law will hold for any unknown chemical compound that may be discovered in the future.

There are several processes too often confounded with induction which should be carefully distinguished from it. (1.) Any logical operation which contains nothing in its conclusion which is not stated in the premises from which it is drawn, *i. e.*, in which the conclusion is not wider than the premises, is not properly induction. Thus were we simply to say that all chemical compounds *which we have examined* combine in definite proportions, this would not be an induction. It would be merely a short way of expressing the several facts already known, that oxygen and hydrogen, etc., combine thus. We perform a process of inductive reasoning only when we infer from the known instances to all instances of chemical combination. (2.) Mathematical proofs, though leading to general propositions, are not inductions, because there is no inference from known to unknown cases. Suppose, for example, that we have proved that the three angles of a triangle are equal to two right angles. This is in reality only shown to be true of the particular triangle before us in our mind, or drawn on paper, but since we see that the same method of proof will apply to any other triangle we assume the theorem to be a general truth. The reasoning is not, "All the triangles that I know have the sum of their angles equal to two right angles, hence I infer that all triangles possess this property," but, "The method of reasoning applied to this particular triangle evidently applies to all possible triangles, and hence the theorem is true of all." Mill suggests that if this is to be allowed the title of induction at all, it should be called *Induction by parity of reasoning*. (3.) Another process often confounded with induction is that which is called by Whewell the *Colligation of Facts*, which is a mere description of observed facts, and not an induction from them. To illustrate, when the astronomer Kepler announced as the result of his observations on the planet Mars that it moves in an ellipse, he simply expressed in that one statement that the successive positions of the planet in the heavens lie in such a curve. There was no inference of unknown from known. Instead of saying that on successive evenings Mars occupied certain definite positions, and stating these in order, he said that these positions are such as would be occupied by a body revolving about the sun in an ellipse of known dimensions and position.

7. Deduction. The general laws of any series of phenomena once obtained by the inductive process, we are frequently enabled to extend our knowledge very greatly by the application of deductive reasoning. For assuming the truth of the law in general, we may calculate what effects should occur under given conditions of its action, and confirm these calculations by direct experiment. Thus, to take a very simple instance, supposing the chemical law of definite proportions to be proved, on finding any new substance we may assume that if its constituents are chemically united, their proportions are the same in all specimens, and if on analyzing different samples of it we find this is not the case, we may justly conclude that the union is not purely chemical, thus obtaining a new fact regarding the substance.

If the results of deduction coincide with those of observation and experiment, the strongest confirmation of the theory in question is produced. In case of a want of coincidence, it does not necessarily follow that the theory is false, inasmuch as there may be certain circumstances which we have neglected to take into account in our calculations.

Deduction often brings facts to our knowledge which it would be very difficult, if not impossible, to ascertain by inductive reasoning alone, since it directs us to the proper course of experimentation in any given case. In the "interrogation of nature" everything depends upon the question put, as the only answer given is, "Yes" or "No." Deduction informs us as to the question which we should ask. As an example in illustration, suppose that we wish to know the maximum amount of work which could be obtained with a perfect steam-engine burning a known quantity of fuel. We are incapable of answering this question by direct experiment until we can construct a perfect engine, and even in that case it would be impossible to prove it to be perfect by inductive methods. If, however, we accept the truth of the law, that a given quantity of fuel when burnt can produce only a definite quantity of mechanical work, we have an immediate answer, since a perfect engine would utilize all the heat produced by the combustion, and the equivalent of this heat expressed in units of mechanical work would evidently be the amount required. Our experimentation should therefore be directed to the obtaining of this equivalent.

From what we have said it will be seen that the process of deduction to be complete must embrace three distinct operations; (1) Induction, (2) Ratiocination, (3) Verification. As the basis there must be some law or laws previously ascertained by induction as general facts from which the unknown particulars may be deduced. Hence to pursue such a deductive investigation we have first to learn the law of each separate cause which contributes toward the unknown effect.[1] Next comes the determination of the effects from a knowledge of the causes, which is an operation of reasoning often of the utmost complication, and which even in some of those cases apparently the most simple, is so baffling that we have as yet obtained only approximate results. Finally, every result of deductive reasoning should be verified by observation or experiment, that we may know whether in our calculations we have neglected to consider any of the attendant circumstances.

---

[1] The case of deduction from hypothesis is considered in § 9.

8. Application of Mathematics. The methods of deduction assume an incalculable importance in Physics, because we are thus enabled to apply the processes of mathematics to determine what will be the result of the action of various concurring forces whose laws have already been ascertained. The results thus obtained will be equally certain with the prior inductions on which the general laws are based; provided, of course, that no elements are omitted from the calculation. It should, however, be observed that the remark in the preceding paragraph concerning approximate solutions frequently applies to this use of mixed mathematics, since even the most subtle processes of the calculus often fail to furnish more than these. Those subjects to which mathematics is thus applied, are frequently known as the *Physico-mathematical Sciences*, and there are now few portions of Physics which have not a more or less developed mathematical theory.

9. Hypothesis. In the absence of certainty as to the causes or mode of production of phenomena, we frequently make use of hypotheses regarding them, and then by the application of deductive processes ascertain whether the result of these calculations coincides with the facts obtained by observation. That is, in the course of reasoning explained in § 7, we substitute a supposition for the first process there mentioned, that of a prior induction. If the supposition with which we start is reasonable, this becomes a valuable addition to our methods for the discovery of truth. But it should be borne in mind that the mere fact of coincidence between observed phenomena and the results deduced from the hypothesis does not necessarily show that the assumed hypothesis is true. To prove this it must also be demonstrated that no other hypothesis will explain the known facts, and unless the supposition is of a nature to allow of such a demonstration, it is not perfectly satisfactory, since it is capable neither of being proved nor disproved. Any hypothesis, however, if plausible, may be of great service as a guide to experimentation.

10. Explanation of Laws. In speaking of physical phenomena we constantly make use of the terms *explanation of phenomena* or *explanation of laws*. It is desirable that a clear conception should be formed of the meaning of these. Any fact is said to be *explained* when the laws of causation on which its production depends are stated. In like manner, a law of nature is explained by showing it to be a particular case of a more general law. There are three methods of explanation of laws which apply in different cases. (1.) The first is that in which several causes unite to produce a joint effect equal to the sum of the effects which each would produce were it acting alone. In this case "the law of complex effect is explained by being resolved into the separate laws of the causes which contribute to it." Thus in the consideration of curvilinear motion we shall explain the law of the movement of a planet about the sun by showing the motion to be the result of the joint action of two forces, one of which is continually drawing the planet towards the sun, while the other continually tends to cause it to fly off in a tangent to its orbit. The orbital motion is thus *explained*, as it is shown to result from forces with whose laws we are already familiar. (2.) "A second case is when between what seemed to be the cause, and what was supposed to be its effect, further observation detects an intermediate link; a fact caused by the antecedent, and in its turn causing the consequent, so that the cause at first assigned is but the remote cause, acting through the intermediate phenomena." The bleaching action of chlorine is explained in this manner. Between that gas and certain bases, specially hydrogen, there is an affinity so great, that chlorine will decompose most compounds of the metallic bases or hydrogen. These are constituent elements of almost all coloring matters, so that the latter are decomposed by chlorine, and their color caused to disappear. Here, the intermediate fact is the affinity of chlorine for hydrogen. In like manner the efficacy of chlorine as a disinfectant is explained, since hydrogen is an essential component of the infectious matter. (3.) The third method of explanation is what has been called the *subsumption* of one law under another. It is "the gathering up of several laws into one more general law which includes them all." As the best possible example of this we cite the generalization of Newton, proving that the force acting to keep the planets in their orbits by tending to draw them towards the sun, and that causing a body to fall to the earth, are both due to an attraction exercised by every particle of matter in the universe upon every other particle.

If the student on reaching that point will carefully read the section of this work treating of the history of the law of universal gravitation, in connection with the present chapter, he will find excellent examples of almost every process which we have described in the preceding pages, as the series of inductions which led to the establishment of that law is the most brilliant in the history of science.

Our explanations, as we call them, give us no insight into the inner causes of phenomena. "What is called explaining one law of nature by another, is but substituting one mystery for another, and does nothing to render the general course of nature other than mysterious; we can no more assign a *why* for the more extensive laws, than for the partial ones. The explanation may substitute a mystery which has become familiar, and has grown to *seem* not mysterious, for one which is still strange. And this is the meaning of explanation in common parlance. But the process with which we are here concerned often does the very contrary; it resolves a phenomenon with which we are familiar, into one of which we previously knew little or nothing, as when the common fact of the fall of heavy bodies is resolved into a tendency of all particles of matter towards one another. It must be kept constantly in view, therefore, that when philosophers speak of explaining any of the phenomena of nature, they always mean (or should mean) pointing out not some more familiar, but merely some more general phenomenon of which it is a partial exemplification, or some laws of causation which produce it by their joint or successive action, and from which, therefore, its conditions may be determined deductively. Every such operation brings us a step nearer towards answering the question, which was stated some time ago as comprehending the whole problem of the investigation of nature, *viz.:* What are the fewest assumptions which being granted, the order of nature as it exists would be the result? What are the fewest general propositions from which all the uniformities existing in nature could be deduced?

"The laws, thus explained or resolved, are sometimes said to be *accounted* *for;* but the expression is incorrect if taken to mean anything more than what has been already stated. In minds not habituated to accurate thinking, there is often a confused notion that the general laws are the *causes* of the partial ones; that the law of general gravitation, for example, causes the phenomenon of the fall of bodies to the earth. But to assert this would be a misuse of the word cause; terrestrial gravity is not an effect of general gravitation, but a *case* of it; that is, one of the particular instances in which that general law obtains.[1] To account for a law of nature means, and can mean, no more than to assign other laws more general, together with collocations, which laws and collocations being supposed, the partial law follows without any additional supposition."[2] The cause of these ultimate laws of action is to be found only in the design of the ruling Mind of the universe.

11. Construction and Correction of Theories. In the mathematical discussion of physical questions, we generally construct our theories by first considering only the simpler and more essential elementary forces acting to produce the phenomena in question. These discussed, other forces which merely modify the general result are taken into account in the form of corrections. To illustrate, suppose we wish to deduce the curve followed by a body, as a cannon-ball, projected into the air, a problem of the greatest importance in the science of gunnery. The consideration of the two principal forces acting upon the ball, viz., the original force of projection, tending to make it proceed in a straight line, and the attraction of the earth, which constantly draws it towards the ground, shows that the path would be a certain mathematical curve known as the *parabola*, were there no disturbing forces.

Such disturbing forces exist, however, in the resisting air through which the projectile moves, so that it is only in a vacuum that the parabolic curve could be strictly followed. The effect of atmospheric resistance in altering the path must then be taken into account as a correction to the original theory, which can be done if we know the relation between the retardation and the velocity of the projectile. Often, as in the case under consideration, these corrections are very difficult to apply, owing to our ignorance of the exact law of the disturbing action, or to complexities in its application when known. In like manner, when considering the theory of machines, we first suppose their component parts to possess no weight, to be absolutely rigid or perfectly flexible, and to act upon each other without friction, and we determine the law of the transmission of power under these conditions. We afterwards ascertain what loss of power there will be on account of friction, and also estimate the effect of the weight of the different parts, and their want of perfect rigidity or flexibility. Not until all these corrections have been applied to the original theory, does it meet the requirements of practical science.

---

[1] It is customary, however, to speak of the fall of bodies as being caused by *gravitation*, meaning by this not the law, which is merely the statement of the general fact, but that force (of the nature of which we are profoundly ignorant) which causes the tendency of particles of matter to approach each other.

[2] Mill, *System of Logic*, Vol. I., Bk. III., p. 529, 6th Ed.

12. Discussion of Observations: Analytical and Graphical Methods.

After making a series of experiments or observations with the intention of determining the law of any phenomena, there are two methods of mathematical discussion that we may apply to them. These are (1) the *Analytical*, and (2) the *Graphical Method.*

Analytical Method. In applying this method, we make use of algebraic symbols, designating each one of the quantities considered by a letter, and discussing their relations by the analytical processes of pure algebra or the calculus.

Graphical Method. This is geometrical, the quantities considered being represented by lines, from whose relations the relation of the quantities themselves is deduced.

The first of these methods is more accurate than the second, but is at the same time more difficult of application owing to accidental errors in observation, and the complexity of many laws. The graphical construction appeals at once to the eye, and is frequently of the greatest service, because of the ease with which the relations among the lines of the figure, and hence among the quantities, are determined.

As a simple illustration of the use of the graphical method let us take the case of a daily record of thermometrical observations extending through a month. We draw two lines, *OX* and *OY*, Fig. 1, at right angles to each other, for purposes of reference. These are known as the *axis of X* and the *axis of Y*, respectively. Their point of intersection *O* is called the *origin of coördinates.* Lines drawn from *OY* parallel to *OX* are called *abscissas*, which are positive to the right, and negative to the left of *OY*, and lines from *OX* parallel to *OY*, *ordinates*, which are positive above, and negative below *OX*. We represent times by abscissas and temperatures by ordinates, laying off on *OX* spaces proportional to the time from the beginning of the observations, and parallel to *OY* spaces proportional to the temperatures at those times. Passing a curve through the points thus determined, we have the desired *thermogram*, or temperature curve. The curve of Fig. 1, is constructed from the data given in the annexed table, and represents the mean daily temperature on the summit of Mt. Washington during the month of January, 1871.

Date.	*Temperature.*	*Date.*	*Temperature.*	*Date.*	*Temperature.*
Jan. 1.	— 3.0	Jan. 11.	21.5	Jan. 21.	10.5
" 2.	8.2	" 12.	32.5	" 22.	— 28.5
" 3.	— 1.6	" 13.	36.0	" 23.	— 2.6
" 4.	— 14.8	" 14.	33.5	" 24.	— 2.5
" 5.	8.7	" 15.	30.2	" 25.	— 15.9
" 6.	12.5	" 16.	31.7	" 26.	3.8
" 7.	— 7.2	" 17.	2.7	" 27.	— 4.5
" 8.	— 5	" 18.	1.5	" 28.	— 5.6
" 9.	— 7.7	" 19.	8.0	" 29.	6.5
" 10.	0.7	" 20.	15.5	" 30.	10.5
				" 31.	27.7

It is convenient to record many continuous phenomena in this manner.

As an application of the graphical method to the discussion of physical laws, we give the following illustration. Let it be required to find the relation between the volume of a quantity of gas and the pressure to which it is subjected. Suppose that experiments have given the results shown in the following table.

Pressure.	*Volume.*	*Pressure.*	*Volume.*	*Pressure.*	*Volume.*	*Pressure.*	*Volume.*
30	100.00	50	60.00	70	42.86	100	30.00
32	93.75	55	54.54	80	37 50	110	27.27
36	83.33	60	50.00	90	33.33	120	25.00
40	75.00	65	46.54				

Denote pressures by abscissas and volumes by ordinates, and construct the curve AC, Fig. 2. This will be found to be an *equilateral hyperbola*, of which OX, OY are asymptotes, a curve having the property that the product of the ordinate at any point by the corresponding abscissa, is a constant. Since the ordinates have been made proportional to the volumes of the gas, and the abscissas proportional to the pressures, it follows that when a gas is compressed, the product of its volume by the pressure upon it is a constant, which is the law sought.

A consideration of Fig. 2 will also show an easy manner of finding the volume of gas corresponding to any pressure not given in the table. Thus if we wish to ascertain the volume under a pressure of 45, we simply lay off $Oa = 45$, and draw ab perpendicular to OX till it meets the curve at b: the line $ab = 66.7$, represents the volume wished, since $Oa \times ab$ is a constant, and Oa has been made proportional to the given pressure. From this example it will be seen that the graphical method gives a simple way of determining intermediate values of a variable, or *interpolating*, as it is called.

LIST OF WORKS OF REFERENCE.

For further information concerning the subjects treated in the preceding pages, the student may consult,

A System of Logic, by John Stuart Mill; Book III., *On Induction.*

Preliminary Discourse on the Study of Natural Philosophy, by Sir J. F. W. Herschel.

Philosophy of the Inductive Sciences, by Wm. Whewell.

History of the Inductive Sciences, by Wm. Whewell.

Cours de Philosophie Positive, par Auguste Comte, *Leçons* 1–39. (*Astronomie, Physique, Chimie.*)

More extended notices of the Analytical and Graphical Methods, and the general discussion of observations will be found in

Lecture Notes on Physics, by Prof. A. M. Mayer (Phila., 1868), pp. 28–49.

Elements of Physical Manipulation, by Prof. E. C. Pickering (Boston, A. A. Kingman, 1872), pp. 1–25.

In these two works a number of valuable references are given to articles bearing upon the subject in question.

CHAPTER II.

WEIGHTS AND MEASURES.—INSTRUMENTS OF MEASUREMENT.

13. In physical investigations it is constantly necessary to determine various magnitudes, so that it will now be advisable to consider the two systems of weights and measures in common use, which are known as the English and the French System.

English System; Linear Measure. The unit of English linear measure is the *yard* of 36 inches, a length which has come down to us from very early times.

The statutory magnitude of the yard is said to have been originally determined in 1120 by the length of the arm of Henry I. There being a great discrepancy among the different yards in use in Great Britain, Graham, in 1742, after a comparison of all the yards and ells of best authority, constructed a standard of which Bird made two exact copies, one in 1758, and a second in 1760. An Act of Parliament in 1824, declared that the length of the yard constructed in 1760, when at a temperature of 62° Fahrenheit, should be considered as the legal standard of linear measure for the kingdom, providing for its recovery in case of loss by making it to consist of 36 inches of *such length* that 39.13929 of them should be equal to the invariable length of the pendulum beating seconds at London, in a vacuum, and at the level of the sea. This Act therefore made the length of the seconds pendulum the basis of the English System. The length of such a pendulum had been carefully measured by Capt. Kater, who determined its value in standard inches of the scale of 1760 as given above. The standard scale, together with that of 1758, was destroyed by fire on the burning of the House of Parliament in 1834, and it became necessary to construct a new one. The scientific committee to whom the matter was referred, did not deem it expedient to adopt the method of restoration by the use of the pendulum, as provided in the Act of 1824, because serious sources of error had been shown to exist in the processes used by Kater in his measurements, rendering it impossible to verify these within $\frac{1}{500}$ of an inch, while a copy of any existing scale could be constructed so as to vary far less than this amount from the original. They therefore advised that all the most authentic copies of the former standard should be compared, and a new one constructed from these, which was accordingly done. They also advised that the new standard should in no way be defined by reference to any natural basis, such as the pendulum, and in the Act of 1855 legalizing the new standard, the yard is defined as being the distance at 62° Fahr., between two marks on a certain standard bar. The British yard, therefore, is no longer based on the length of the seconds pendulum at London, but is simply a copy of a material scale kept in the Exchequer Office at Westminster.

The unit of linear measure adopted by the United States is essentially identical with that of Great Britain. The standard at the Office of Weights and Measures at Washington is a brass scale. 82 inches in length, constructed by Troughton for the Coast Survey.

The distance from the 27th to the 63d division of this at 62° Fahr., is equal to one U. S. yard. Two copies of the new British standard, one of bronze and one of iron, presented to the United States by the English government, compared with the Troughton scale, show that the U. S. standard exceeds the present British one by 0.00087 in.

The units for the measure of surface and solidity, as the square and cubic foot or yard, are obtained directly from the linear unit.

14. Weight. The unit of weight at present adopted by Great Britain is the ***Avoirdupois pound*** of 7000 grains, the standard being a platinum weight kept in the Office of the Exchequer.

The oldest English legal standard known was the *Tower pound*, as it was called, which was used even before the Norman Conquest. and was somewhat lighter than the present *Troy pound* of 5760 grains. It was employed at the Mint as the standard for coinage up to the time of Henry VIII., by whom it was abolished in 1527, though the Troy pound was more or less in use, at least as early as the reign of Henry V. (1413–1422). There was also at this time another pound in general use, called the *libra mercatoria*, weighing about 6750 grains. When the Avoirdupois pound was first introduced is not definitely known, but it appears in statutes of the time of Edward III. (1327–1377), and like the Troy pound was probably introduced by the Lombard bankers of that day. It seems to have derived originally from a Greek weight, the *mina*, of 6945.3 grains, which the Romans subdivided into 16 ounces, while the Troy pound came from the Roman weight of 5759.2 grains, the $\frac{1}{125}$ part of the *Talent of Alexandria*. which they divided into 12 ounces. During the reign of Elizabeth (in 1582 and 1588) Avoirdupois weights were deposited in the Exchequer, but the Troy pound continued to be the legal standard.

All the early weights were carefully examined by Graham in 1742, and in 1758 a Parliamentary Committee recommended the construction of a new standard Troy pound, by which all other weights should be regulated. The bill carrying these recommendations into full effect was never passed, but three one-pound weights were made, one of which was deposited in the Parliament House. In 1818. the question was revived, and a committee, among whose members were Young, Wollaston and Kater, was appointed to investigate the subject. They advised that the Troy pound of 1758 should be the standard for the kingdom, and that the Avoirdupois pound, which had never been legally defined, should be declared to contain 7000 grains Troy. They also defined the grain weight by adopting the results of Shuckburgh and Kater, according to which a cubic inch of distilled water, weighed in air by brass weights at a temperature of 62° Fahr., with the barometer at 30 inches, was equal to 252.458 grains, of which 5760 constituted a Troy pound. In case of loss, the standard pound was to be restored by reference to the weight of a cubic inch of water. Their report became a law in 1824, but only ten years later the standard pound of 1758 was destroyed by fire. In 1838, Baily, Sir J. F. W. Herschel and others were requested to take into consideration some means for the restoration of the lost standards. In 1841, they reported adversely to restoring the pound in the manner designated in the Act of 1824, owing to the great discrepancy existing among various determinations of the weight of the cubic inch of water, which rendered the accuracy of Shuckburgh's results very questionable. In fact, the weights assigned to a cubic inch of distilled water by different experimenters exhibit a variation amounting to $\frac{1}{1200}$ of the total amount, while a copy of any existing standard could be made which should not vary from the original by more than $\frac{1}{1000000}$ of its actual weight. The committee therefore advised that the new standard be made by a comparison of the best authentic copies of the old standard. They also recommended that the Avoirdupois pound be made the legal standard rather than the Troy pound, since the former was in much more general use. The construction of the new standard was undertaken in 1843 by Prof. W. H. Miller, who collected thirteen authentic copies of the original Troy pound of 1758. Eleven of these were finally rejected from consideration, as they were made of brass and were considerably oxydized, and the two remaining, which were of platinum, were the only ones used in the determination of the exact weight of the standard of 1758. From measurements of these, a Troy pound of 5760 grains was constructed of platinum, which was regarded as identical with the former standard, and from this a platinum standard equal to $\frac{7000}{5760}$ of the weight of the Troy pound was made, together with four copies. This was constituted the legal standard Avoirdupois pound of Great Britain by the *Weight and Measure Act* of 1855. Experiments made upon the actual standard when completed, showed it to really weigh 7000.00093 grains, of which the old standard contained 5760.

The United States Standard is the Troy pound of the U. S. Mint, copied by Capt. Kater from the British Troy pound of 1758. An Act of Congress of 1834, defines it as being equal in weight to 22.794422 cubic inches of distilled water at its maximum density, but the incorrectness of the weight of a cubic inch of water under those circumstances, which was assumed by that act, causes the definition of the actual pound weight to be in error.

15. Capacity Measures. The British unit for all measures of capacity is the *Imperial gallon*, which contains 10 Avoirdupois pounds of distilled water at a temperature of 62° Fahr., with a barometric height of 30 inches. According to the best determinations, it contains 277.123 cu. in. under those circumstances. The Act of 1824 stated its contents as 277.274 cubic inches, which has since been shown to be an incorrect value. The U. S. Standard is the *wine gallon* for liquids, and the *Winchester bushel* for dry measure, derived from the English measures of the same name.

The former measure is defined by Act of Congress as containing 231 cubic inches, the latter 2150.42 cubic inches at 39° Fahr., 30 inches barometer. But unfortunately the capacity of the *British* Winchester bushel is 2150.4 cubic inches at 62° Fahr., so that at our standard temperature it contains only 2148.9 cubic inches, and at no common temperature are the two bushels equal. The same is the case with the wine gallon. The British Imperial bushel, a measure in common use, but not a legal standard, contains 8 Imperial gallons or 2216.984 cubic inches at 62° Fahr.

The method of ascertaining the number of cubic inches contained in the standard gallon or bushel, is by finding the weight of the quantity of distilled water necessary to fill it under given circumstances of temperature and atmospheric pressure. This weight divided by the weight of a cubic inch of water, gives the capacity. This method evidently requires a precise knowledge of the weight of a cubic inch of water, which, as we have stated, is so difficult of exact determination that somewhat different results have been obtained by the most skilful experimenters, causing a slight variation of the actual from the legal contents of the U. S. standards. The greatest possible deviation of these from the statutory capacity is, however, so small as not to be of very great practical consequence, the gallon and bushel never being used in any but commercial measurements.

There has been even more variation in the values of the various capacity measures of past times than among weights. The capacity of the ordinary wine gallon has gradually risen from 216 cubic inches, its contents in 1299, to its present value of 231 cubic inches. The Winchester bushel has had equally variable values, having risen from 2114.68 cubic inches in 1266, to 2150.4 cubic inches, its present capacity. The British Imperial standard gallon is a unique measure, and is derived directly from the Avoirdupois pound, while the other measures were originally obtained from the Tower and Troy pounds.

16. Circular or Angular Measure. The unit of angular measure is the *degree*, which is the $\frac{1}{360}$ of a circumference. It is subdivided into 60 *minutes*, and each of these minutes into 60 *seconds*, smaller angles being indicated by decimals of a second.

Until within two or three centuries it was customary to carry this sexagesimal subdivision still farther, each second consisting of 60 *thirds*, the third, of 60 *fourths*, and so on, a cumbersome method, which has happily gone entirely out of use.

17. French or Metric System. History. This system was adopted in France in 1795, during the Revolution, and was intended to furnish a uniform system of weights and measures, all of which should be based upon a single linear unit, and the various subdivisions of which should be purely decimal.

As early as 1790, Talleyrand proposed the establishment of a general system, in the construction of which all the civilized nations of the world should be invited to join, and suggested the pendulum beating seconds as an invariable standard from which to derive the linear unit. In accordance with the proposition, a party of five [1] from the Academy of Sciences was appointed to investigate the subject. This committee reported in 1791, advising that the project should be carried out, but preferred as a standard of reference the quadrant of the earth's meridian included between the equator and the pole, which was then supposed to have the same value in all longitudes, while the seconds pendulum was known to vary in length with the locality. They also suggested the employment of the pendulum as a secondary standard, from which to recover the other in case of its loss. In accordance with their recommendation, which was immediately transmitted to the Assembly, Délambre and Méchain were deputed to ascertain the length of a quadrant of the meridian, and for this purpose measured a meridional arc of about 9 1-2 degrees, passing through France from Dunkirk to Barcelona. This work occupied a number of years, but the Assembly made use of a value of the quadrant given by some earlier measurements, and promulgated the system in 1793, not adopting the present nomenclature, however, until 1795. At length, in 1799, an international convention was assembled at Paris, which finally decided the values of the various standards of weights and measures.

---

[1] Borda, Lagrange, Laplace, Monge and Condorcet.

18. Linear Measure. The French arc, compared with arcs previously measured in Peru and Lapland, gave as the distance from the equator to the pole, 5,130,740 *toises*, the toise being an old French measure equal to about 2.132 English yards. It was decided that $\frac{1}{10000000}$ of this distance should be called a *metre*, and made the basis of the new system. The unit adopted in Délambre and Méchain's measurements was the *Toise of Peru*, the same that had been used in the earlier measurements of the Peruvian arc by Bouguer and La Condamine. The Act of 1799 decrees the metre to be $\frac{443296}{864000}$ of the length of the standard *Toise de Pérou*, which is an iron bar, correct at 62.25 degrees Fahr., made in 1735 under the direction of Godin. A standard metre of platinum was made from this by Lénoir. The value of the metre in English inches is 39.370432.[1]

More extensive geodetic measurements made during the present century show that the value of the quadrant obtained by the French astronomers is too small. Bessel gave as its true value 10,000,856 metres. Moreover, owing to the recently demonstrated fact that the earth is not an oblate spheroid, but more nearly an ellipsoid with three unequal axes, it follows that the value of the quadrant of a meridian varies with the longitude. Hence the basis of the French system is in reality as much a local one as is the pendulum. It should also be borne in mind that from the terms of the law of 1799 the French metre, like the present English foot, is simply the length of a legalized material scale.

The metric system is purely decimal. The larger divisions of linear measure are the *myriametre* = 10,000 metres, the *kilometre* = 1,000 metres, the *hectometre* = 100 metres, and the *dekametre* = 10 metres. Those less than 1 metre are the *decimetre* = $\frac{1}{10}$ metre, the *centimetre* = $\frac{1}{100}$ metre, and the *millimetre* = $\frac{1}{1000}$ metre.

Measures of surface and solidity are formed from those of linear dimensions, precisely as in the English system.

19. Weight. The unit of weight is the *gramme*, which was ordered by statute to be the weight of a cubic centimetre of pure water in vacuo, at 39.1 degrees Fahr. The actual standard platinum weight (by Fortin) of 1000 grammes, in the French Archives, is a little greater than it should be according to this law. The gramme is divided into *decigrammes*, *centigrammes* and *milligrammes*, equal to $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ grammes respectively. The larger weights are the *dekagramme* = 10 grammes, the *hectogramme* = 100 grammes, and the *kilogramme* = 1000 grammes. The weight of the standard kilogramme above mentioned is 15432.34874 grains, or 2.204,621,250 lbs. Avoirdupois, which gives 15.43234874 grains as the weight of the gramme.[1]

---

[1] According to the measurements of Capt. Clarke in 1866. Two other measurements of the metre have been made; that of Kater (1821), which gave as a result 39.37079 in., and that of Hassler (1832), giving 39.3810327 in. The latter is the value which has been used by the U. S. Coast Survey. The results of Capt. Clarke are those now in use by the British Ordnance Department.

20. Capacity Measure. The unit of capacity measure is the *litre*, which is the volume of a kilogramme of pure water at 39.1 degrees Fahr., and 30 inches barometer. It was originally intended to be a cubic decimetre, but is somewhat larger, owing to the excess of weight of the standard kilogramme mentioned above. It is equal to 61.02499 cubic inches.[2]

21. Circular Measure. The French also applied the decimal system to angular measure, and a number of the circles used in the observations for determining the length of the French meridional arc were thus divided. Each quadrant contained 100 *grades or degrees*, which were again subdivided into tenths and hundredths. Hence the centesimal degree was equal to 0.9 of a sexagesimal degree. As the sexagesimal method of division was universally employed long before the metric system originated, the French division of the quadrant has never been extensively used, even by the nation which devised it.

22. Full tables of all French weights and measures employed in physical operations, together with their values in English units, will be found in the appendix to this volume.

Owing to its great simplicity, the metric system is gradually displacing all others. At the present time it is the most widely used of any, and is rapidly gaining in general favor, most of the European nations having adopted it either wholly or in part. Its employment in all commercial transactions was legalized by Great Britain, in 1864, and by the United States, in 1866[3].

Sir John F. W. Herschel has proposed the polar axis of the earth as a more logical basis than any other for a system of weights and measures, since, unlike the pendulum and the quadrant of latitude, it is not local in its nature. While this is very true, the wide acceptance of the metric system renders it undesirable to attempt a change which, even were it possible, would be chiefly of theoretical advantage. What is most desirable among different nations is some system, at once uniform and simple, both of which requisites are already supplied; the derivation of the unit is a matter of but slight practical importance. Another advantage urged by Herschel, is, however, that if the present English inch were increased by $\frac{1}{1000}$ of its total value, the polar axis of the earth would contain just 500,500,000 such inches; and that 1728 such cubic inches of water would weigh 1000 ounces Avoirdupois if the present value of the ounce were increased by $\frac{1}{18}$ of a grain. On the other hand, it is not probable that the most accurate value that we possess of the earth's polar axis is sufficiently near to absolute correctness to warrant a change at present.

---

[1] Miller.

[2] Clarke.

[3] The student may be interested to know that the U.S. five-cent piece is 2 centimetres in diameter, and weighs 5 grammes.

23. Unit of Time. The unit employed in physical observations extending over a considerable time, is the *mean solar day* of 24 hours, that being the mean interval between two successive transits of the sun across the celestial meridian.

The mean interval is taken because the time elapsing from day to day between the passages of the sun across the meridian is not constant, owing to the facts that the earth's motion about the sun is not uniform, and the sun's apparent path is not in the celestial equator, but inclined to it at an angle of 23° 28′. Our ordinary clocks are adjusted to keep mean solar time. Astronomical clocks, on the contrary, are adjusted to *sidereal* or *star-time*. A sidereal day is the interval between two successive transits of the same star across the meridian,[1] and is the exact period of the revolution of the earth on its axis. It is 3 m. 56.5 s. shorter than the mean solar day. The sidereal day, like the solar, is divided into hours, minutes and seconds. When astronomical clocks are used for the estimation of time in physical experiments, it is customary to reduce their indications to mean solar time by a simple proportion. For small intervals of time the unit in common use is the *second*.

Until recently it was supposed that the period of the earth's rotation had not varied by $\frac{1}{100000000}$ of its length since 720 B. C., but an error was found by Adams in the work of Laplace, on whose calculations the above supposition was based, and he has shown that the time of revolution is diminishing at the rate of 22 s. in a century.

24. Instruments of Measurement. Physical measurements are of five kinds, according to the nature of the magnitude to be determined. Instruments for precise measurements may be classified according to their use, as follows:—

1. Instruments for the measurement of angles.
2. " " " length.
3. " " " volume.
4. " " " weight.
5. " " " time.

It will be most convenient to consider the construction of the majority of these instruments as they present themselves in the sequel, in connection with the subjects of base lines, determination of the standards of capacity measures, balances, etc., but we shall here describe a few which are of very general application. These are the *graduated circle* for angular measurement, the *vernier* and the *cathetometer*, an instrument for the precise measurement of vertical distances.

---

[1] More strictly, of the *vernal equinox*, but the difference is only one one-hundred and twentieth of a second in a day.

25. Instrument for measuring Angles. Angles are generally measured by circles divided at the circumference into degrees and fractions of a degree. The common *Engineer's Transit*, Fig. 3, will give a general idea of an instrument for measuring angles. It is designed to measure either vertical or horizontal angles, and therefore carries two circles, one vertical, the other horizontal, which can be seen at *C* and *D*. At *AB* is a telescope moving upon the same horizontal axis with *D*, while the whole instrument turns horizontally about an axis attached to the centre of *C*. Screws, *S* (*leveling screws*), are attached, by which the circle *C* can be made exactly horizontal, also screws *E* and *F* called *tangent screws*, by which a slow motion can be given to *C* or *D*. At *G* is seen a compass needle, which is sometimes of service, and at *HK* a level attached to *AB*, so that the telescope may be made horizontal if necessary. The telescope *AB*, is furnished with two cross-hairs in its eye-piece, at right angles to each other (Fig. 4). Suppose it is necessary to measure the vertical angle between two points. The observer places his eye at the telescope, focuses it on one of the points, bringing the intersection of the cross-hairs to coincide with the image of that point, and observes the reading on the vertical circle given by the index *I*. He then turns *AB* about its axis until the second point comes into the field of view, and fixes the intersection of the cross-hairs upon it in the same manner as before. Then taking the new reading given by *I*, the difference between it and the former reading is the desired angle. As the telescope moves horizontally about the centre of *C*, it evidently makes no difference whether the two points are or are not in the same vertical. Horizontal angles are measured in a similar manner by means of the horizontal circle *C*. There is another index opposite *I*, and in delicate measurements the mean of the readings of both should be taken, which will eliminate any error from eccentricity in the mounting of the divided circles.

26. Vernier. It is often necessary to measure lines and angles more closely than it is convenient to divide the scale or circles used. In such cases we may estimate fractions of one division by the eye, especially if aided by a magnifying glass, with considerable accuracy. Various other methods have been devised, one of the simplest and most generally applicable of which is the *Vernier*, so called from its inventor, Pierre Vernier, of Brussels. (1631.) To understand its construction, let *AB*, Fig. 5, be an enlarged representation of a scale divided into millimetres, with which we wish to read to tenths of a millimetre. Suppose that a scale *DC* is made by taking 9 divisions of *AB* (in the figure 110 to 119), and dividing this length into 10 equal parts, numbered from 0 to 10. Each division of *DC* will equal $\frac{9}{10}$ of a millimetre, and will be less than one division of *AB* by $\frac{1}{10}$ mm., so that if the 0 of *DC*, which is the vernier scale, be made to coincide with any division of *AB*, as the 750th, division 1 of *DC* will fall short of 751 of *AB* by $\frac{1}{10}$ mm., 2 will fall short of 752 by $\frac{2}{10}$ mm., 3 will fall short of 753 by $\frac{3}{10}$ mm., and so on, until the 10th division of *DC*, which coincides with 759 of *AB*. To comprehend the method of using it, let it be required to find the distance of some point *S*, Fig. 6, from any other point. The scale *AB* is laid so that its 0 shall be opposite one of the points, and the vernier is slid along *AB* until its 0 denoted by the arrow rests against *S*. The distance from the other point as indicated by the scale, is evidently 751 and a fraction millimetres, the whole number being given by *AB*. The fraction is obtained from *CD*, by counting its divisions upward from 0, until one is reached which coincides with one of the divisions of *AB*. The number of this vernier-division gives the fractional reading, which in this case is the number of tenths of a millimetre ($\frac{3}{10}$) between 751 and the 0 line of the vernier. For as each division of *CD* is less than one division of *AB* by $\frac{1}{10}$ mm., if the coincidence takes place as in the figure at the 3d vernier-division above 0, the space between 0 and 751 must equal 3 scale-divisions minus 3 vernier-divisions, that is $\frac{30}{10}$ mm. — $\frac{27}{10}$ mm. = $\frac{3}{10}$ mm. Hence in the instance supposed, the total distance between the two points is 751.3 mm.

From this it will be seen that in general, if we call n the number of divisions of the scale equal in length to $n + 1$ divisions of the vernier, in which case $\frac{n}{n+1}$ is the value of one division of the latter, and denote by x the distance between that scale-division which coincides with a vernier-division, and the scale-division next below the 0 of the vernier, the fractional reading is $x - \frac{nx}{n+1} = \frac{x}{n+1}$. (1.)

Verniers are applied to graduated scales and circles of all kinds, and furnish a ready means of increasing their delicacy of measurement. There are various modifications in use for particular purposes, but the principle of construction is the same in all. The smallness of the fractions which are indicated by any vernier depends upon the number of its divisions. Thus if, as in Fig. 7, 9 scale-divisions are sub-divided into 10 parts, it reads to $\frac{1}{10}$ mm. If 99 scale divisions were sub-divided into 100 parts, it would read to $\frac{1}{100}$ mm., since each vernier-division would then = $\frac{99}{100}$ mm., and one scale-division minus one vernier-division = $\frac{100}{100}$ mm. — $\frac{99}{100}$ mm. = $\frac{1}{100}$ mm. Or, generally, if x in Eq. (1) is made equal to unity, $\frac{1}{n+1}$ will be the minimum limit of reading in terms of the length of the divisions of the scale. Hence to find the limits of reading of any vernier, *divide the length of one scale division by the number of divisions of the vernier.*

27. Cathetometer. In many physical investigations it is necessary to measure the difference between the vertical heights of two or more objects to which it is difficult to apply an ordinary scale. In such cases we make use of an instrument invented by Dulong and Petit, and known as the *Cathetometer* (Fig. 7). The essential parts of this instrument are a heavy vertical pillar of metal *AB*, mounted upon a base *HK*, which is furnished with leveling screws. The pillar *AB* carries a scale *FG*, divided into millimetres. A telescope *CD* is furnished with cross-hairs, and mounted horizontally upon a frame *TN*, which can be raised or lowered by sliding it along *AB*. A screw *S* serves to give a very slow vertical motion to the telescope when desired. The frame *TN* carries a vernier *E*, and a level *L* is attached to *CD* to detect any deviation from horizontality. To measure the vertical distance between any two points, after having carefully levelled the instrument, the frame *TN* is moved until one of them lies in the axis of the telescope, which is focussed carefully, and its cross-hairs are made to coincide exactly with the point by means of the screw *S*. The reading of the scale and vernier is then taken. This done, the telescope is raised or lowered until the other point coincides with the cross-hairs, when the scale and vernier are again read. The difference between these two readings is the vertical distance between the points. That the cathetometer may give accurate results it is evident that the axis *AB* must be exactly vertical, and that the line of sight of the telescope must remain parallel to itself when raised or lowered.

28. Measure of Time. Measurements of considerable intervals of time are made with clocks or chronometers. The measurement of very minute intervals is accomplished by various forms of instruments called *chronographs*. Many of these will be described in succeeding chapters.

REFERENCES.

For additional information the student may consult

Account of the Proportions of French Measures and Weights, from the Standards of the same kept at the Royal Society, by G. Graham; *Philosophical Transactions*, Vol. XLII., p. 185.

Account of Comparison of Standard Yards and several Weights with original Standard, by G. Graham; *Phil. Trans.*, Vol. XLII., p. 541.

Construction of the new Imperial Standard Pound and its Copies of Platinum, by W. R. Miller; *Phil. Trans.*, Vol. CXLVI., p. 753.

Construction of the new Standard of Length, by G. B. Airy; *Phil. Trans.*, Vol. CXLVII., p. 621. (In this article will be found an extensive list of references to articles bearing on the subject of which it treats.)

On the Correct Adjustment of Chemical Weights, by Wm. Crookes; *Chemical News*, Apr. 19, 1867.

Comparison of Standards of Length of England, France, Belgium, etc., by Capt. A. R. Clarke; (London, Ordnance Survey Department) pp. 105, 163.

Tables, Meteorological and Physical, prepared for the Smithsonian Institution, by Arnold Guyot (Washington, 1858).

The Yard, the Pendulum and the Metre, a Lecture by J. F. W. Herschel, published in *Familar Lectures on Scientific Subjects* (London, A. Strahan & Co., 1867), p. 419.

The Metric System, by Charles Davies, (New York and Chicago, A. S. Barnes & Co., 1871) contains the lecture cited in the preceding reference, together with the *Report on Weights and Measures*, by John Quincy Adams, (1821) *and the Report of the University Convocation of the State of New York*, (1870).

The Metric System, with Appendix on Capacity Measures, by F. A. P. Barnard. (New York, 1872.)

Base du System Métrique, Tome III, par Jean B. J. Delambre et Pierre F. A. Méchain.

Recueil Officiel des Ordonnances et Instructions sur la Fabrication et la Verification des Poids et Mesures. (Paris, 1839.)

Atlas des Poids et Mesures dressé en Execution de l'Ordonnance Royale. (Paris, 1839.)

Rapport sur les Poids et Mesures envoyés au Gouverment des Etats-Unis, par A. Vattemare. (Measures executed by I. T. Silbermann.)

Rapport sur la Révision des Etalons en 1867 *et* 1868; *Annales du Conservatoire des Arts et Métiers*, Tome IX., No. 33, 1er Fascicule, p. 1.

Also consult references under the chapter of this work treating of the *Pendulum.*

The following *Parliamentary Sessional Papers* may be consulted for records of the legislation of Great Britain relative to the subject of Weights and Measures: —

On Weights and Measures; Parl. Papers, Reports of Committees, 1813–14, Vol. III., No. 290. Also *Rep. Committees, Jan. to July*, 1821, Vol. IV., No. 571; and *Report of Commissioners*, by Clark, Gilbert, Wollaston, Young and Kater, No. 383 of same volume.

Minutes of Evidence on Weight and Measure Bill; Rep. Committees, 1824, Vol. IV., No. 94. *Also Rep. Committees*, 1834, Vol. XIV., No. 464.

Report on Weights and Measures, and Minutes of Evidence; Rep. Committees, 1835, Vol. XIV., No. 292.

Report of Commissioners appointed to consider the Steps to be taken for the Restoration of the Standards of Weight and Measure, by Airy, Baily, Bethune, Herschel, Lefevre, Lubbock, Peacock and Sheepshanks; *Rep. Commissioners*, 1834–5, Vol. I., No. 177.

Report of Commissioners appointed to consider Steps for the Restoration of Standards; Rep. Commissioners, 1842, Vol. XI., No. 356.

Report of Committee appointed to superintend the Construction of the Parliamentary Standards, by Airy, Rosse, Wrottesley, Lefevre, Lubbock, Peacock, Sheepshanks and Miller; *Rep. Commissioners*, 1854, Vol. XIX.

Report of Committee recommending the Legalization of the Metric System, together with Minutes of Evidence; Rep. Committees, 1862, Vol. VII., No. 411.

Bill authorizing the Use of the Metric System; Parl. Bills, 1864, Vol. IV., Nos. 24, 165.

Report on the Exchequer Standards of Weight and Measure, by H. W. Chisholm, with *Notes* by G. B. Airy; *Accounts and Papers*, 1864, Vol. LVIII., No. 115.

Bill to amend acts relative to Standard Weights and Measures; Public Bills, 1866, Vol. v., No. 166.

First Report of Warden of Standards, 1866–7; *Rep. Commissioners*, 1867, Vol. xix.

Report of Commissioners appointed to inquire into the condition of the Exchequer Standards, and on the Abolition of Troy Weight, with Minutes of Evidence; Rep. Commissioners, 1870, Vol. xxvii.

On the application of the Metric System to India; Rep. Commissioners, 1870, Vol. liii., No. 225.

The legislation of the United States relative to Weights and Measures, and the reports to the Government on the construction of standards, may be found in the various *Congressional Documents*, as follows:—

Report of Committee on Fixing Standards of Weight and Measure; Reports of Committees, 15th *Congress*, 2d *Session* (1818–19), Vol. vi., Doc. 109.

Report on Weights and Measures, by J. Q. Adams, Secy. of State (Feb. 22, 1821); *Executive Papers*, 16th *Cong.*, 2d *Sess.*, Doc. 109.

Report of Committee to whom was referred the Report of the Secy. of State; Reports of Committees, 17th *Cong.*, 1st *Sess.* (1821–2), Vol. ii., Doc. 65.

Comparison of Weights and Measures of Length and Capacity, by F. R. Hassler; *Exec. Doc.* 22d *Congr.*, 1st *Sess.*, (1832.) Vol. vi., Doc. 299.

Construction of Standards of Weights and Measures; Reports of Committees, House Rep., 23d *Congr.*, 2d *Sess.*, (1835) Vol. i., Doc. 132.

Letter from F. R. Hassler, *in Rep., of Secy. Treas.; Exec. Papers., House Rep.*, 24th *Congr.*, 1st *Sess.*, (1835) Vol. ii., Doc. 32.

Report on Furnishing States with Standards; Rep. Comm., House Rep., 24th *Congr.*, 1st *Sess.*, (1836) Vol., i., Doc. 259.

Ibid., Vol. ii., Doc. 449.

Report on Progress of Construction of Standards, by F. R. Hassler; *Exec. Papers*, 25th *Congr.*, 2d *Sess.*, (1837–8) Vol. xi., Doc. 14.

Report on Construction and Completion of Standards for all the States of the Union, by F. R. Hassler, (July 4th, 1838); *Ibid.*, Doc. 454.

Reports by F. R. Hassler; *Senate Doc.*, 26th *Congr.*, 1st *Sess.*, (1839–40), Vol. ii., Doc. 15; Vol. viii., Doc. 608. *Also Sen. Doc.*, 26th *Congr.*, 2d *Sess.*, (1840–1) Vol. xi., Doc. 20.

Report on Progress in Construction of Standards of Liquid and Capacity Measure, by F. R. Hassler; *Sen. Doc.*, 27th *Congr.*, 2d *Sess.*, (1841–2) Vol. iii., Doc. 225.

Reports by F. R. Hassler; *Sen. Doc.*, 27th *Congr.*, 3d *Sess.*, (1842–3) Vol. ii., Doc. 11.

Report relative to Weights, Measures and Balances, by F. R. Hassler; *Exec. Doc.*, 28th *Congr.*, 1st *Sess.*, (1843–4) Vol. iv., Doc. 94.

Reports on Weights and Measures, by A. D. Bache; *Exec. Doc.*, 28th *Congr.*, 2d *Sess.*, (1844–5) Vol. iv., Doc. 159.

Exec. Doc., 29th *Congr.*, 1st *Sess.*, (1845–6) Vol. vii., Doc. 225. *Exec. Doc.*, 30th *Congr.*, 1st *Sess.*, (1847–8) Vol. ix., Doc. 84.

Report on Construction and Distribution of Weights, Measures and Balances, and on Comparison of Foreign Standards, by A. D. Bache; *Exec. Doc.*, 34th *Congr.*, 3d *Sess.*, (1856–7) Vol. vi., Doc. 27.

Report on Weights, Measures and Balances, by A. D. Bache; *Exec. Doc.*, 35th *Congr.*, 2d. *Sess.*, (1858–9) Vol. vi., Doc. 6. (This report contains a resumé of the work relative to the construction of standards previous to

1858. Also the titles of all State Acts bearing on the subject from 1819 to 1854.)

Report of Committee appointed for Purpose of investigating the Metric System, and accompanying Bill; Acts and Resolutions of 39th *Congr.*, 1st *Sess.* (1865-6), p. 350.

For a detailed account of the various forms of *Vernier* in practical use, see *A Treatise on Land Surveying*, by W. M. Gillespie. (New York, Appleton & Co., 1867.)

For methods of graduating scales consult

Gasometry, by Robert Bunsen, trans. by H. E. Roscoe (London, Walton & Maberly, 1857); p. 25.

Traité élémentaire de Physique, par P. A. Daguin; Tome I., p. 22.

Cours de Physique de l'Ecole Polytechnique, par M. J. Jamin; Tome I., p. 25.

Edinburgh Encyclopedia, Article *Graduation*, by Troughton.

CHAPTER III.

PROPERTIES OF MATTER.

29. Matter.—General Properties. Whatever can be perceived by the ordinary operations of the senses is *matter*. It possesses two essential characteristics: (1) *extension*, or the property of occupying space, and (2) *impenetrability*, by virtue of which no two bodies can occupy the same space at the same time.

The second of these sometimes seems to be contradicted by experience. Thus when an inverted bottle is immersed in water the liquid rises to some distance inside; but in this case the air is not penetrated, it is merely compressed, so that a portion of the space which it formerly possessed is occupied by the water. A stone sunk in a vessel of water does not penetrate it, but displaces a quantity equal to its own bulk. And in all other cases in which at first sight one body seems to penetrate another, it will be found that there is really a displacement of matter.

Both the above characteristics are essential to the existence of matter. A *shadow* occupies space, but is not matter as it is not impenetrable.

30. Three States of Matter. Matter exists in three different conditions, which are known as the *solid*, *liquid* and *gaseous* states. Solids are distinguished by possessing a definite form which resists any change with considerable force. Liquids take the form of the vessel containing them, their particles not being firmly united like those of solids, but moving upon each other with the greatest ease. Gases possess this mobility of particles in common with liquids, but in addition to this their particles have a constant tendency to separate from each other, so that some ex terior effort, such as the resistance of the walls of the vessel containing them, is necessary to limit their volume. This expansive tendency may be shown by placing a bladder partly filled with air in a receiver in which we can produce a vacuum by means of an air-pump. When the external air is removed the bladder gradually swells out, owing to the expansion of the air which it contains on the removal of the external atmospheric pressure, which previously kept it under a more limited volume. As examples of the three states of matter, we may mention iron, wood and stone, among solids; water, alcohol and mercury, among liquids; and air, carbonic acid gas and common illuminating gas, among gases.

Liquids and gases are often collectively called *fluids.*

The same substance may exist in each of the three states of matter according to the external circumstances. Thus ice when heated assumes the liquid form of water, and if this is heated still more it finally assumes a gaseous state by passing into steam. Sulphur also assumes the solid, liquid and gaseous forms, successively, when heated, and many substances, as carbonic acid, laughing gas, etc., which are gaseous at ordinary temperatures, can be made to assume a liquid, and finally a solid form under the influence of great pressure and extreme cold.

31. Particular Properties of Matter.—Divisibility.

In addition to the preceding there are certain particular properties which are common to all kinds of matter, though not essential to our idea of its existence.

(1.) Matter is *divisible.* We know of nothing which can not be separated into parts, these again into smaller parts, and so on. The extent to which matter is divisible will be appreciated from the following examples. Gold leaf has been hammered into leaves but $\frac{1}{25000}$ mm. in thickness. Silver wire gilded on the exterior has been drawn out to such a degree of fineness that the coating of gold was only $\frac{1}{800000}$ mm. thick. The silver core was then dissolved away by acid, leaving a gold tube of this excessive thinness. With a microscope it is possible to see a particle $\frac{1}{4000}$ mm. in diameter, so that we are able to divide gold into particles $\frac{1}{4000}$ mm. in diameter, and $\frac{1}{800000}$ mm. in thickness, each of which possesses all the qualities of the metal. Platinum wire has been drawn so fine that it took 200 metres of it to weigh 1 centigramme, its diameter being but $\frac{1}{1200}$ mm. This was done by enclosing a platinum wire in a closely-fitting silver tube, drawing out the compound wire thus made, and then removing the silver coating with acid. The thickness of a soap bubble just before bursting is but $\frac{1}{1000000}$ mm. A gramme of carmine will tinge 60,000 gr. of water, or about 9,000,000 drops, so that in one of these drops there is but $\frac{1}{90000000}$ grammes of coloring material. By optical methods it is possible to detect even $\frac{1}{3000000000}$ of a gramme of soda. It is said that a single grain of musk has perfumed a room 12 feet square for several years without sensible loss of weight; and an extract of spirit of musk has given a distinct odor to 2,000,000,000 times its weight of a certain liquid. These examples might be multiplied indefinitely, but we will mention only a single additional one, drawn from the organic world. Certain microscopic plants, the *Diatoms*, are covered with a siliceous shell, weighing but $\frac{1}{3000000000}$ of a gramme. On this shell we can see striæ not over $\frac{1}{3400}$ mm. in width and thickness, thus perceiving a portion of siliceous matter, whose weight would be only about $\frac{1}{7900000000000}$ of a gramme.

32. Atoms and Molecules. The question here arises whether this division could be carried on indefinitely. Early in the history of science this was a disputed subject. Anaxagoras and Aristotle held matter to be infinitely divisible, while Leucippus taught that there were ultimate indivisible particles, to which Democritus gave the name of *atoms*. (B. C. 500.) The atomic doctrine was also taught by Epicurus. In modern times Descartes rejected the theory, while Gassendi maintained it. Up to a comparatively recent time these opinions were mere philosophical speculations resting upon no solid basis, but in later years certain laws of chemical combination, crystallography and molecular physics, have shown it to be highly probable that matter is composed of ultimate particles of inconceivable, though finite minuteness, which are physically incapable of further subdivision by any known process. These atoms are grouped together to form larger masses, called *molecules*, which may again be arranged in still more complex aggregations.

The molecules of a body are separated from each other by spaces called *pores*, which are much greater in size than the atoms themselves. Most bodies are composed of compound molecules formed by the chemical union of atoms of unlike nature to form a new substance. Hence it is convenient to distinguish between *integrant* and *constituent molecules*, constituent molecules being aggregations of similar atoms, while integrant molecules are formed by the union of the constituent molecules of dissimilar substances. Thus marble is composed of integrant molecules of carbonate of lime, while each of these integrant particles is, in its turn, composed of constituent molecules of calcium, carbon and oxygen. The exceeding minuteness of molecules renders it impossible to obtain any direct measurement of their size, but Sir Wm. Thomson has shown[1] that the mean distance between the centres of contiguous molecules of matter is probably less than $\frac{1}{100000000}$ and greater than $\frac{1}{2000000000}$ of a centimetre.

33. Compressibility. All substances are *compressible*. On the application of external pressure to any body, its volume is diminished. This can only arise from the closer approximation of its molecules, which renders it evident that these are not in absolute contact. The compression of solids is noticeable in any structure supporting a great weight. The stone pillars sustaining the dome of the Pantheon at Paris, were sensibly compressed when that structure was erected upon them. In the operation of coining, the metal check upon which the impression is struck is noticeably diminished in volume. The compressibility of liquids is less evident, and until within a century they were quite generally believed to be absolutely incompressible. In 1762, Canton first showed this idea to be erroneous, and found that water was compressed about $\frac{46}{1000000}$ of its volume by a pressure of 1 atmosphere (1033.6 grammes to a square centimetre). More accurate experiments since that time have confirmed this general result. Gases are by far the most easily compressed of all substances. This may be shown by means of an air-tight piston moving in a glass tube closed at one end. The piston can be forced into the cylinder, compressing the air contained in it. On withdrawing the pressure the gas expands, forcing the piston back until it has attained its original volume. This last effect is due to the perfect elasticity of the gas, a force tending to make it return to its primitive volume. The same result occurs in the case of a liquid, and in a less degree with solids, the elasticity of the latter class of substances being in general very imperfect when compared with that of the two former.

---

[1] Article in *Nature*, Mar. 31. 1870, p. 551.

34. Expansibility. On the application of stretching forces to bodies, or on heating them, their volume is found to increase. The expansion of solids by external forces may be shown by stretching a wire or an India-rubber tube. Expansion by heat can be rendered evident by means of a ball of iron, made of such size as to just pass through a ring of the same material when cold. If heated the ball will no longer do this, on account of its increased volume. If a quantity of liquid be contained in a tube with a bulb at one end, its level will be seen to rise in the tube when heated, and if such a bulb-tube contain air, or any other gas, a drop of liquid being placed in the tube to serve as an index, the expansion of the gas is so great that it will be apparent when the bulb is simply clasped by the hand.

35. Porosity. The fact that the molecules of any substance are not in contact, which we have already noticed, is shown to be true by the phenomena of compressibility and expansibility. The name *pores* is given to these intermolecular spaces. The porosity of substances can be proved directly in various ways. An experiment celebrated in the history of science is that first performed by Lord Bacon, and afterwards by the Academy of Florence (1661). Wishing to determine whether water was compressible, they filled a hollow silver sphere full of that liquid, and after closing it tightly flattened it under a screw-press. This operation of course diminished the capacity of the sphere. At the close of the experiment the exterior was found to be covered with a fine dew, showing that the water had been forced through the pores of the silver. The Academicians drew from this two conclusions: first, that the pores of silver were larger than the molecules of water, which is true, and secondly, that water was absolutely incompressible, which is erroneous. Other examples occur in the arts. Thus the manufacture of steel from wrought iron depends upon the absorption of a portion of carbon by the metal. It has been shown that platinum and cast iron, when heated to redness, are porous to gases. The porosity of liquids and gases is seen in the fact that in most chemical combinations the volume of the resulting compound is less than the sum of the volumes of the components. The same is true in the case of many solutions. Thus anhydrous alcohol and water mixed in the proportions of 116 to 100, sustains a diminution in volume equal to 3.7 per cent., and with other proportions there is still a diminution, though not as much as with those mentioned. The same thing occurs on mixing water and sulphuric acid. The porosity of liquids is farther shown by their ability to absorb gases. Thus water at 60° Fahr. will absorb about 720 times its bulk of ammonia gas, the resulting increase of bulk being only 50 per cent. In the case of gases, similar contractions are observed. Thus 4 volumes of nitrogen unite with 2 volumes of oxygen to form 4 volumes of protoxide of nitrogen, the resulting compound having a volume of only two-thirds that of its constituents when in an uncombined state.

In all these cases it will be seen that the particles of one kind of matter enter the pores of another, but do not penetrate the atoms, so that the statements already made relative to the impenetrability of matter are in no way invalidated.

We should carefully distinguish between the intermolecular pores, of which we have been speaking, and the larger interstices existing among the particles of solid bodies to which the name pores is also applied. These last, often called *organic* or *structural pores*, according as they occur in organic or inorganic bodies, are so large as readily to be seen with the aid of a microscope, or even with the unaided eye. Such pores may be seen in wood, and in many minerals. By slight pressure, mercury can be forced through the organic pores of a piece of wood, and *hydrophane*, a kind of agate, is opaque when dry, but on immersion in water absorbs that liquid, increasing in weight, and at the same time becoming translucent. The disintegration of rocks by frost is an effect due to the structural pores. In the wet weather of autumn, water enters them which in winter is frozen, and by its expansion causes the rock to crumble.

36. Mobility. Constant experience shows us that matter has the property of *mobility*, or capability of motion from place to place.

37. Indestructibility. Matter is *indestructible.* However greatly the external appearance of a body may be changed, not the smallest particle is actually destroyed. Thus, when a piece of wood is burnt, the only visible remainder is a small quantity of ashes, yet were we to collect the carbonic acid and aqueous vapor formed during the combustion, and weigh them, the sum of their weights, together with that of the ashes, would exactly equal the original weight of the wood. The same is true in all other cases in which matter is seemingly destroyed, owing to its assuming an invisible state.

38. Ether. The phenomena of light, heat and electricity, have led to the supposition that in addition to the three ordinary states of matter there is a fourth condition in which it may exist, and that such matter, known as the *ether*, is universally distributed, penetrating the pores of all bodies, and diffused throughout all space, existing even in the most perfect vacuum. The majority of scientists regard this ether as a substance distinct from all others, though some maintain with much plausibility that it is merely ordinary matter in a very tenuous state.[1] Ether has frequently been called an *imponderable* fluid, owing to an idea once very generally entertained that it possessed absolutely no weight. It is more probable, however, that it is ponderable, like all other matter, the impossibility of our weighing it proceeding from the fact that we cannot render any substance devoid of it, a necessary condition for ascertaining its weight.

39. Forces. The various changes which are constantly taking place in matter invariably appear to us as forms of motion, either of the body as a whole, or of its molecules. Any change of place is evidently a motion as a whole, and we shall see hereafter that all other changes, as, for example, those of temperature, are forms of molecular motion. In general, whatever produces the effect we denominate a *force*, so that we may also define a force as anything that produces, or tends to produce, motion. That the actual production of motion does not necessarily follow upon the action of a force, arises from the fact that the tendency to motion thus impressed is resisted by some other force opposed to the first. Thus, a body suspended upon a spring-balance tends to fall to the earth because of its weight; but it does not fall, since this downward force stretches the spring and develops in it a certain amount of tension, which is exactly equal and opposite to the weight of the body.

The forces acting upon matter are of two kinds, *attractive* and *repulsive forces*, the first tending to make particles or masses of matter approach, the second tending to cause them to recede from each other. We may also classify forces according as they act at sensible distances and between masses, or only at insensible distances and between molecules. The latter are denominated *molecular forces.* The principal forces acting at sensible distances, *i. e.*, at distances exceeding $\frac{1}{100}$ of a millimetre, are the attraction of *gravitation*, and the attractive and repulsive forces of *magnetism* and *electricity*, while those acting at insensible distances are *cohesion*, *adhesion*, *capillarity* and *chemical affinity*, together with the repulsive force due to *heat*, and certain other molecular repulsions. Gravitation acts among all bodies, causing them to tend towards one another; the forces of electricity and magnetism are manifested only under peculiar circumstances, and may be either attractive or repulsive. Cohesion acts to unite particles of the same kind of matter in a mechanical union; thus the molecules of a piece of iron are held together by this force. Adhesion causes a mechanical union of particles of different kinds of matter, as the particles of two pieces of wood are united by glue. Capillarity is a manifestation of force developed by the united action of adhesion and cohesion; and chemical affinity causes the chemical union of unlike particles of matter to form an entirely new substance. These, together with the repulsive forces which we have mentioned, will be discussed in detail hereafter.

---

[1] See *An Essay on the Correlation of Physical Forces*, by W. R. Grove (New York, Appleton & Co., 1865), E. L. Youmans, Editor; p. 183, *et seq.*

The attractive forces acting among the molecules of a body would cause them to approach and come into absolute contact, were it not for the coincident action of a molecular repulsive force. This just balances the molecular attraction, so that the particles are kept at a certain distance apart. If they be forced nearer together the repulsive force is increased, and they tend to separate, while if the distance between them is increased by any external force, the molecular attractions cause them to tend to approach and assume their original position.

40. Polarity. Certain bodies have the property of exerting different forces at opposite ends, so that a body which is attracted by one end will be repelled by the other. The simplest illustration of this action is the case of two magnets. Suppose them to be balanced so as to move freely about a vertical axis; then, as is well known, one end of each will point towards the north. Denote by A the end of each, which assumes this position, and by B the end of each pointing toward the south. Then it will be found that if the A end of one magnet be presented to the B end of the other, an attraction will be exerted between them, while if the A end of one be approached to the A end of the second, or if the two B ends be placed near together, they will repel each other.

REFERENCES.

Size of Molecules, by Sir Wm. Thomson; *Nature*, Vol. I., p. 521.
Letter on the Size of Molecules, by Sir Wm. Thomson, read before the

Manchester Lit. and Phil. Soc., Mar. 22, 1870. Published in *Nature*, Vol. II., p. 56.

Nature of Ether. See *Essay on the Correlation of the Physical Forces*, by W. R. Grove. (*Youman's Compilation.*)

CHAPTER IV.

FORCE AND ITS MEASURE.—LAWS OF MOTION.

41. Motion. Motion is a progressive change of position in space. We can be acquainted with none but *relative* motions, for we have no means of ascertaining the fact that a body really changes its position except by comparison with some other body not affected with the same movement. To know the *absolute* motions of any body this point of comparison must be absolutely at rest. But we can find no such point. The earth and other planets have a double motion about their axes and around the sun, the sun moves on its own axis, and also probably in an orbit around one of the fixed stars. Hence terrestrial bodies can only be at rest relatively to each other and to the earth, and as we must refer their motions to bodies which are moving themselves, the absolute motion of any point can not be determined.

Moreover, an apparent condition of rest of a body as a whole, is compatible with movements of great energy among the particles of which it is composed. We shall see as we proceed that we have reason to believe that the molecules of all substances are in a constant state of vibration to and fro; these vibrations being so minute and rapid that we can not perceive their existence except by their effects. There is no such thing, then, as absolute rest, and when the term rest is used it is to be understood as relative.

42. Velocity. The rapidity of motion is measured by the *velocity*, which is the linear space passed over in a unit of time. Thus a body moving over 10 metres in one second is said to have a velocity of 10.

43. Force.—Pressures and Impulses. A *force*, as already stated, is any cause tending to produce or modify motion. *Mechanics* is the name given to that branch of physics which treats of the laws of force in general.

Mechanical forces are commonly divided into *pressures* and *impulses*, according as the time of their duration is sensible or insensible in magnitude. Thus the weight of a body resting upon a surface is an example of a pressure, while the blow of a hammer in driving a nail is an example of an impulse. In the case of the supported weight the pressure tends to produce motion, but this tendency is resisted by the reaction of the surface on which the body rests. If this be removed motion ensues, and the body falls. Evidently the tendency of an impulse may be resisted in a similar manner.

The difference between these forms of force, however, is only one of degree and not of kind, for impulses are, in reality, only pressures acting during a very short time. The impulse given to an arrow by a bow, for example, is due to the continued action of the bow-string for a fraction of a second. The impulse given to a cannon-ball is caused by the pressure of the gases formed by the combustion of the gunpowder during the time that the ball occupies in passing through the whole length of the bore. So when motion is destroyed, as in the case of a cannon-shot fired into a wall, the body is not brought to rest immediately, but penetrates a certain distance, in proportion to the magnitude of its moving force. Since an impulse is a pressure acting during an infinitely short time, it follows that any pressure may be considered as caused by a succession of impulses repeated at infinitely small intervals.

44. Measure of Pressures. In order to estimate the magnitude of a pressure we make use of instruments called *dynamometers.* The common spring-balance (*Leroy's dynamometer*), Fig. 8, is an example. It consists of a helical steel spring fixed in a frame, and connected with an index moving over a graduated scale. On the application of a force at *B* the spring is coiled more closely, and from the amount of this coiling, as shown by the index *I*, the force is estimated. Another form of dynamometer is shown in Fig. 9. The arcs *AC*, *BD*, are connected with the arms of a steel spring *E*, which is bent more closely together in proportion as the weight suspended from *C* is greater. The magnitude of the force is read by the scale upon *BD*. Various other forms of dynamometers are used, varying in construction according to the nature and magnitude of the force to be measured. The graduation of the scales of all these instruments is performed by experiment, known weights being applied, and the corresponding positions of the index marked. If the pressure is so exerted as to cause motion, as in the case of a horse drawing a canal-boat, or a locomotive moving a train of cars, the resistance or pressure produced can be measured by interposing a dynanometer in the line of its action.

45. Measure of Moving Forces. In many cases, especially where the moving forces are impulses, it is impossible to measure them in the manner described above, and hence it is important to have some other means of estimating their magnitude. A method by which we may compare all forces is by their effect in producing or destroying motion. If, for example, we see a man throw a ball with a velocity of 5, while another man throws the same ball with a velocity of 10, the time of action of the arm upon the ball being the same in both cases, we infer that the second man exerts twice as much force as the first. This is merely reasoning from the effect to the cause, measuring the latter by the former, on the principle that the cause must be proportional to the effect.

It will here be necessary to define two terms which we shall frequently employ. The *mass* of a body is the quantity of matter that it contains. Equal masses contain equal quantities of matter. *Acceleration* is the velocity generated in a body by the action of a force during a unit of time.

The measurement of forces by the motion which they produce depends upon the following proposition, which is verified by universal experience.

(I.) *Constant forces are proportional to the products of the masses on which they act, by the accelerations impressed upon those masses.*

Hence if F, F', be two forces, and M, M', masses upon which they impress accelerations a, a', respectively,

$$F : F' :: Ma : M'a'. \quad (2)$$

Thus if two forces acting for 1 second generate velocities 1 and 2 in bodies whose masses are 3 and 5, respectively, the forces are to each other as $1 \times 3 : 2 \times 5$, or as $3 : 10$, that is, $F : F' :: 1 \times 3 : 2 \times 5 :: 3 : 10$.

If in (2) we suppose the masses of the bodies to be the same, $M = M'$, and $F : F' :: Ma : Ma'$, or

$$F : F' :: a : a' \quad (3); \qquad \text{that is,}$$

(II.) *Constant forces are to each other as the accelerations which they impress on equal masses.*

Thus if two forces acting for the same time communicate velocities 3 and 5 to a body, they are to each other as $3 : 5$, that is, $F : F' :: 3 : 5$.

If in (2) the accelerations a, a' are equal, $F : F' :: Ma : M'a$ or,

$$F : F' :: M : M' \quad (4); \qquad \text{that is,}$$

(III.) *Constant forces are to each other as the masses on which they impress equal accelerations.*

Hence if the forces F, F', acting during a unit of time, communicate equal velocities to masses 8 and 15, the forces are to each other as $8 : 15$, that is, $F : F' :: 8 : 15$.

The preceding propositions are of fundamental importance, and should be thoroughly mastered by the student before proceeding farther.

46. Momentum. The product of the massa of a body and its velocity is known as its *momentum* or *force of motion*. The momentum of a body is therefore proportional to the quantity of matter which it contains, and also to the velocity with which it moves.

If bodies whose masses are as $3 : 5$ have the same velocity, their momenta are as $3 : 5$; if the bodies are equal, and their velocities 2 and 3 respectively, the momenta are as 2 : 3; and if the masses are 3 and 5, and their velocities 2 and 3 respectively, the momenta are as $3 \times 2 : 5 \times 3$, or as 6 : 15.

The fact that momentum depends on both velocity and mass, is susceptible of numerous illustrations. A small cannon-ball moving with a high velocity is capable of doing an immense amount of damage, though its mass is comparatively small, since its rapid motion gives it a great momentum. A large ship, on the other hand, though progressing with an almost imperceptible velocity, will overcome a great resistance. Thus strong cables are sometimes snapped when the vessel scarcely seems to move, its great mass making up for its small velocity. So a slowly-moving iceberg coming in collision with a ship may destroy it.

47. Momentum a Measure of Force.

Proportion (2) may be stated thus: *Forces are proportional to the momenta generated by their constant and uniform action during a unit of time, or during equal times.* The momentum impressed upon a body by any force acting for a unit of time is, then, a measure of that force.

The momentum which can be generated by the action of any force during a given time is evidently a constant quantity, whether the body acted upon be large or small. Hence if a large body is put in motion by a given force it receives a proportionally less velocity than one of smaller size. In fact, if the force F will cause accelerations a, a', in bodies of mass M, M', respectively, the momenta generated are Ma and $M'a'$. As these are equal, $Ma = M'a'$, whence

$$a : a' :: M' : M \quad (5); \qquad \text{that is}$$

the velocities[1] *impressed upon two bodies by the action of the same or equal forces during equal times are inversely as their masses.*

If a mass is already in motion, and a force acts in opposition to it, the momentum destroyed by the force is equal to that which it would generate in the same time by its action on that mass when at rest. Forces are therefore proportional to the momenta destroyed by them in equal times.

In all the preceding discussion two things have been assumed; first, that the action of the forces is *constant* and *uniform*; and, secondly, that *they act during equal times.* Any variation in the intensity of the force acting would of course render our demonstrations invalid, because we should, in reality, be comparing different forces at each moment. The forces compared must also act during equal times, because it is obvious that the longer a force acts the greater the velocity it will produce.

48. Measure of Mass.

We have already defined the mass of a body as the quantity of matter contained in it. The question now arises, how shall we measure this? One method is furnished by the proportions just demonstrated. Since the momentum which a given force can generate in a unit of time is a constant, and the force is therefore measured by this momentum, by choosing a suitable unit of mass,[1] we may write $F = Ma$; whence $M = \frac{F}{a}$ (6). The mass of a body may then be expressed by *the constant ratio between any force and the acceleration which it will produce when acting upon that body.*

---

[1] We can evidently write for a, a', the velocities generated by the action of the forces for a unit of time, any velocities whatever, by making our unit of time larger or smaller, as the case may be.

It will be shown in a succeeding chapter that any body when allowed to fall freely under the influence of gravity acquires a velocity of 9.8 metres per second, a quantity which is usually designated by the letter g. The force causing it to descend is evidently its weight W. Here a force W generates an acceleration g, and as forces are proportional to their accelerations (3), $F : W :: a : g$ (7), whence $\frac{F}{a} = \frac{W}{g}$, or $M = \frac{W}{g}$ (8). The mass of any body is therefore its weight divided by the velocity which it would acquire by falling freely for one second. This quotient is a constant, for if W varies from any cause, g, the acceleration produced by it, must also vary in the same ratio (3), so that the value of $\frac{W}{g}$ will remain unchanged.

This mode of indicating the mass of a body is the one which has usually been followed in modern treatises upon the subject, but some inconveniences, the nature of which we shall presently explain, have more recently led to the use of another method.[2]

The weight of a body may be considered under two aspects: First, as denoting the *force* with which it is drawn towards the earth; and secondly, as denoting the mass of the body as compared with the mass of an arbitrary standard, such as the gramme or pound.

Now if by the weight of a body stated in grammes or pounds, we understand the force with which it is attracted towards the earth, that is if we consider the standard unit of weight to be a unit of force, we cannot express the mass of a body in grammes or pounds, because, as we shall see hereafter, the weight of a given quantity of matter, and hence the quantity of matter contained in a given weight is not the same at all places upon the earth's surface, so that a body of constant mass will possess different weights at the equator and poles. But if by the weight thus expressed we understand merely the quantity of matter contained in the body, compared with the quantity of matter contained in the arbitrary standard of weight of the French or English systems;

---

[1] For the momentum Ma being *proportional* to the force F, the latter must equal a *constant* multiple of the momentum, whatever be the unit of mass chosen, and if this unit be taken so that when $F = 1$ and $a = 1$, M also equals 1, or $F = Ma$.

[2] See *A Treatise on Natural Philosophy*, by Thomson and Tait, Vol. I., p. 166.

that is, if by a gramme or a pound we understand a quantity o matter equal in mass to the standard gramme or pound, we can express the mass of a body in such units, which, though arbitrary, are yet definite and invariable. In this sense it is perfectly correct to speak of the mass of a body as being so many grammes.

Now in common life this use of weight as an estimate of mass is far more general than its use as an estimate of force. In strict language, to be sure, weight is the downward tendency of a body, but we ordinarily employ weights "for the purpose of measuring out a definite quantity of matter; not an amount of matter which shall be attracted to the earth with a given force." Hence our standards of weight are primarily intended as units for measuring mass, and it is a secondary application by which we use them to estimate forces. The latter method of measuring mass is therefore far more simple than the former.

49. Unit of Mass. According to the first method of measurement the unit of mass is the mass of g times the standard unit of weight. For if the unit of mass is so chosen that $M = \frac{W}{g}$, it is evident that if $M = 1$, W must be numerically equal to g. Hence the unit of mass is the mass of g grammes, or g pounds of matter.

The objection to this mode of estimating mass is now apparent. Since the quantity g is a variable, the unit of mass is also a variable quantity, and it is exceedingly desirable to have some unit which shall be invariable. This is furnished in the second method of estimating mass, as in that case the standard gramme or pound is taken as the unit. This gives us an absolute unit of mass, which can be obtained in no other manner. This unit was first brought into general use by Gauss.

50. Unit of Force. The unit of force is the force which, by acting upon a unit of mass for a unit of time, generates a unit of velocity.

The unit of force, according to the first system, is the gramme in French, the pound in English measures. For the unit of mass in that system is g times the unit of weight, and the force with which this is attracted towards the earth is g grammes or g pounds, which in 1 second generates a velocity of g feet, or metres. Hence, as forces are proportional to the accelerations which they produce in equal masses, the force which would generate an acceleration of 1 unit is equal to $\frac{1}{g}$ of that which generates an acceleration of g units, that is, to 1 gramme, or 1 pound. This is called the *gravitation unit of force*, and the system of measurement of masses and forces derived from it is called the *gravitation system* of measurement.

According to the second of these methods of measurement, the unit of mass being a standard gramme or pound, the unit of force is the force which, acting upon a national standard unit of mass for a unit of time, generates a unit of velocity. This unit of force is numerically equal to the unit of mass divided by the acceleration of gravity, that is, to $\frac{1}{g}$ grammes, or $\frac{1}{g}$ pounds, the value of g being the acceleration at Paris or at London, according to the system used. It is known as Gauss' *absolute unit of force*, and the system of measurement of masses and forces based upon it as the *absolute system* of measurement.

To obtain a clearer idea of the value of the absolute unit of force, we must know the numerical value of g. The value of g at London is 32.1889 ft.; and at Paris 9.8087 metres. Hence the British absolute unit of force is equal to the weight of $\frac{1}{32.1889}$ lbs. at London, and the French absolute unit is equal to the weight of $\frac{1}{9.8087}$ grammes at Paris. That is, in round numbers, the British absolute unit of force is equal to the weight of about half an ounce, and the French unit to about the weight of $\frac{1}{9}$ of a gramme.

We may evidently define the French unit of force as *that force which, by acting upon a mass of* 1 *gramme for* 1 *second, generates a velocity of* 1 *metre*, and the British unit as *that force which, by acting apon a mass of* 1 *pound for* 1 *second, generates a velocity of* 1 *foot.*

To transform forces expressed in gravitation units to absolute units, we must evidently multiply their numerical value by g. Thus a force of 10 grammes is equal to 10 × 9.8087 French absolute units of force, and a force of 10 lbs. is equal to 10 × 32.1889 British absolute units.

The student will notice that the equation $W = Mg$, which according to the first system of mass measurement denotes the weight of a body of mass M, in the second system denotes the number of absolute units of force in the downward tendency W, caused by gravity.

51. Representation of Forces by Right Lines. In many problems it is convenient to use a graphical method of representing forces. A force is defined when its magnitude, direction and point of application are given. Hence we may represent the relative magnitude of forces by straight lines, whose lengths bear the same relation to each other as the numerical values of the forces themselves, while the directions of these lines may indicate the direction of the forces, and the point from which the lines are drawn the point of application. Thus two forces of 1 and 3 kgr., applied at a single point and inclined 20° to each other, would be represented by lines AB, AC, Fig. 10, of lengths 1 and 3 units respectively, drawn from a point A, and making an angle $BAC = 20°$.

52 Three Laws of Motion. Law I. The elementary principles relating to the phenomena of motion and force were arranged by Newton under the laws often known as the *Newtonian Laws of Motion.*

LAW I. *A body at rest continues in that state, and a body in motion proceeds uniformly in a straight line, unless acted upon by some external force.*

This law is directly deducible from our fundamental ideas regarding cause and effect. Since every effect requires some cause to produce it, it follows that a body unaffected by any force must remain in the state in which it already exists. A body at rest has no power to put itself in motion; a body in motion has no power to bring itself to rest, or to deviate from its path, but must continue to move in a fixed direction.

The law is also verified by all experience. The incapacity of bodies at rest to change their condition is so obvious as to need no illustration. But the case when they are in motion is not always so clear. Thus a body thrown horizontally does not move uniformly in the line of projection, but proceeds with a gradually diminishing horizontal velocity, until it comes to rest on the ground. This decay of motion seems to contradict the second portion of the law. But a closer examination shows that there is here a force acting upon the body constantly tending to destroy its motion, viz., the resistance of the air; while another force, the attraction of gravitation, continually draws it towards the earth. Again, a ball rolled upon a road does not move on indefinitely, but soon slackens its velocity, and is finally brought to rest. This is no more an exception to the law, however, than the other case, for it is the friction of the ball upon the road that causes its loss of motion. If the friction is diminished the distance to which the ball will roll is proportionally increased: on a common road it stops very soon, on a smooth bowling-green it rolls much farther, while on a sheet of clear ice it goes a very long distance before stopping. In cases where the friction and resistance of the air are reduced to a minimum, we obtain a very long continuance of motion. Thus a nicely-balanced wheel, moving on fine, well-oiled bearings, on being set in rotation will revolve for a very long while. Again, an ordinary clock-pendulum, if detached from the rest of the works and swung, will come to rest after a short interval, but some pendulums of very delicate construction, moving with exceedingly little friction will swing in the air for nine or ten hours, and in a vacuum for twenty-four hours. Since, then, whenever a diminution of motion occurs, we are able to find forces causing it, and since the condition of uniformity of motion is approached in proportion as we do away with these forces, we are justified in concluding that could we eliminate them altogether the loss of motion would entirely disappear. So in all cases of motion in curved lines, as in those just cited of the wheel and pendulum, we find forces at work which deviate the body from the rectilinear path it would otherwise pursue. We therefore conclude that the law, as stated, is true in all cases.

Still further, the consequences deduced from other laws of matter, supposing this law to be true, agree with the results of observation, which could not be the case if the law were false. Thus in Astronomy, the methods used in predicting eclipses assume that the motion of the earth and moon is not altered except by the action of some external force, and as the observed and computed times agree, we are justified in asserting the truth of our proposition. The same remark applies to the remaining laws of motion.

53. Resisting Medium in Space. The question will be asked here, "Is there any example of permanent motion in nature?" The revolution of the planets around the sun offers the nearest approach to perpetual motion with which we are acquainted, but even in this case there is evidence of the existence of a resisting medium acting to slacken their velocities. This medium is so light that its effects are only perceptible in the retardation of a single comet (Encke's), but if it exists it must nevertheless act in the same manner upon all bodies circulating about the sun. The reality of the existence of a resisting medium, though generally believed, is discredited by as high an authority as Sir J. F. W. Herschel, who explains the retardation of Encke's comet in a different manner.[1]

54. Phenomena illustrating the First Law of Motion. The truth expressed in Law I. is the principle of the *Inertia of Matter*. The term *inertia* in strict language means simply the inability of matter to change its state except under the action of some force. It is also universally used in a somewhat different, though analogous sense, as denoting *that property of matter because of which a definite force is necessary to produce a given change in the existing state of a mass*, which is simply another mode of expressing the general idea of momentum.

By means of the first law of motion in connection with the principles of momentum, numerous familiar phenomena are readily explained. The apparent resistance experienced on setting a body in motion, or on stopping a moving body, is a consequence of its inertia. A person riding on horseback at a rapid rate is thrown forward if the horse suddenly stops, because the momentum of his own body carries him onward in the direction in which he was moving. The same thing occurs to the passengers in a railway train stopped by a sudden application of the brakes. If, on the contrary, the car is started quickly they are thrown backward, the motion of the car not being immediately communicated to them. It is because of the inertia of matter that the effects of the collision of trains of cars are so terrible. The locomotive is suddenly brought to rest, while the cars continue to move, thereby piling one upon another, and causing a general destruction. For a like reason when a vessel moving at full speed strikes a sunken reef, the spars upon her deck and even the sailors are sometimes shot violently forward over the bow.

---

[1] See Herschel's *Outlines of Astronomy*, 11th Ed., 1871, § § 577, 570.

"Coursing owes all its interest to the intuitive consciousness of the nature of inertia which seems to govern the measures of the hare. The greyhound is a comparatively heavy body moving at the same or greater speed in pursuit. The hare *doubles*, that is, suddenly changes the direction of her course, and turns back at an oblique angle with the direction in which she had been running. The greyhound, unable to resist the tendency of its body to persevere in the rapid motion it had acquired, is urged forward many yards before it is able to check its speed and return to the pursuit. Meanwhile the hare is gaining ground in the other direction, so. that the animals are at a considerable distance asunder when the pursuit is recommenced. In this way a hare, though much less fleet than a greyhound, will often escape it." [1]

We see a practical application of the first law of motion in the method often used of fixing an axe or hammer firmly on its handle. The tool being placed in a vertical position, with the head uppermost, is moved rapidly downward, so that the end of the handle strikes against some solid object. The motion of the handle is stopped, while the head moves on, thus fixing itself firmly.

The principle under consideration may also be illustrated by a very simple experiment. Let a smooth card be balanced on the tip of one of the fingers. On this place a somewhat heavy coin. If, now,, a quick, horizontal blow be given to the edge of the card with the fore-finger of the other hand, it flies from under the coin, leaving this poised upon the finger. The slight force of friction exerted by the moving card upon the coin, is not sufficient during the short time of its action to impress the motion upon the latter. If the card be pushed slowly the coin moves with it.

55. Time required to produce Change of State in a Body. From these examples, especially the last, we see that if a force acts upon only a few points of a body, a sensible time is required to transmit its effects to every portion of the mass. Several curious phenomena are explained by this fact. A light stick supported by placing its ends upon the edges of two wine glasses will be broken by a quick blow, without affecting its supports, while a less sudden stroke will be liable to break them. In the former case the stick is snapped before sufficient time has elapsed for the blow to be communicated to the glasses. A rifle ball fired through a pane of glass cuts out a round hole, because it progresses too rapidly to allow the motion to be impressed upon any of the particles of the glass, except those immediately in its path, before it has passed entirely through. A slowly-moving bullet, or a stone thrown against the pane, cracks it in every direction. For the same reason the greatest damage in naval combats is caused by shot moving with a comparatively slow motion, as they cause far more splintering of the timbers than a swifter projectile. In like manner a cannon ball may be fired through a partly open door, scarcely moving it on its hinges. A candle fired from a musket will penetrate a board without being greatly crushed. A certain time is required to change the form of a body by crushing, and as this is greater than the time required to overcome the cohesion of the fibres of wood composing the board, the candle goes through before mnch change in its form can take place.

It is because of the time necessary to generate motion in a body that when a horse attempts to start a heavy load by a sudden pull, some part of the harness is liable to give way, while the load remains unmoved, though a slow and steady pull would have put it in motion. In the case of long trains of heavily loaded cars, a great gain ensues from the slight movement allowed by the couplings which unite them to one another. That the train may be set in motion it is necessary to overcome the friction of all the wheels resting upon the rails. The locomotive easily exerts force enough to put the first car in motion, as this can be fairly started before the second is moved at all, because of the slackness of the connecting coupling, and so on for each car in succession. The train is thus put in motion in parts, the inertia of the cars already moving aiding the engine in starting the remainder. Were the whole train rigidly connected, it would be impossible for an ordinary locomotive to cause it to move. When a person falls from a considerable height upon a rock he is severely hurt, but if upon a soft bed he is uninjured, because in the latter case the stopping of the motion is performed gradually, so that the violent shock otherwise produced is obviated.

---

[1] Lardner, *Treatise on Mechanics*, p. 39.

56. Matter being purely passive, it follows that if any force, however small, act upon a body, it must produce a proportionate amount of motion, supposing the body to be absolutely free to move. In practical cases there is a certain resistance to motion offered by friction and other causes, which must be overcome before movement can take place. The least force in excess of this will start the body. Hence it is a general law that in the absence of resisting actions *the smallest force is capable of moving the largest body*. As an illustration of this it is stated[1] that in calm weather and smooth water a very large ship can gradually be put in motion by the efforts of a child pulling at a rope attached to the bow. The effect of the force in producing momentum is of course exactly the same as if it were expended in giving a rapid motion to a small body.

57. Resistance of Media. The resistance offered by a fluid to the motion of a body is due to the pushing aside of its particles by the moving mass, which thus loses a portion of its momentum. The denser the fluid, that is, the heavier its particles, the more resistance it offers to movement in it. Thus a delicate pendulum swung in the air will oscillate for some hours before being brought to rest, while if swung in water it will move for only a few minutes.

58. Law II. *If several forces act upon a body simultaneously, each one of these produces the same effect in magnitude and direction as if it acted alone. This principle is known as the Law of the Independence of Motions.*

The truth of the second law of motion may not appear at first sight. The effect of mechanical forces is to produce motion along their lines of action, and the question will be asked, "How can several motions in different directions coëxist in a body?" The difficulty disappears, however, if it is recollected that all movement in a body is estimated by reference to some point not possessing the same motion. The *absolute* motion of a mass can evidently have but a single direction, but *relatively* to certain points a body may have several simultaneous motions. For example, a man travelling from southeast to northwest is at the same time moving toward the north and towards the west, so that after the lapse of an interval of time he will have passed over a certain number of miles in a northerly, and a certain other number of miles in a westerly direction.

---

[1] Leslie, *Elements of Natural Philosophy*; Vol. I., p. 30.

59. Illustrations of Second Law of Motion. Various illustrations may be cited in proof of this law. The muscular exertion put forth in walking a given distance upon the deck of a steamer is the same, whether the boat is at rest or in motion. If the vessel moves at the rate of five miles an hour with reference to some point on the shore, and a person walks over the deck in the same direction and at the same rate, his velocity relative to that point will be ten miles an hour, and relative to the vessel five miles an hour. If, on the contrary, he walks in an opposite direction, his velocity relative to the vessel will be five miles, but relatively to the shore he will be at rest. Again, imagine a boy sliding upon a moving cake of ice in a direction at right-angles to its line of progression. The boy will be affected with both motions, moving down the stream as rapidly as if he were not sliding, and sliding as rapidly as if the cake of ice were at rest. So two persons standing on the deck of a steadily-moving vessel, can toss a ball to each other as readily as if the boat were at rest. To take the illustration first used by Galileo, a person on shipboard can write with ease, while in his pen coexist the motions caused by the hand, the swaying of the ship and its progressive motion, and the axial and orbital revolutions of the earth. But as the paper possesses all these movements except the first, the letters are traced exactly as if only this motion existed. A ball dropped from the mast-head of a moving ship will strike the deck at the foot of the mast, because the forward motion possessed by it in common with the vessel, is in no way diminished by the downward motion communicated by gravity. A heavy body falling from a balloon which is moving rapidly in a horizontal direction, "during its descent partakes of the balloon's motion, and until it reaches the earth is always seen perpendicularly under the car." A body dropped from the summit of a lofty tower, or allowed to fall down a deep mine-shaft, strikes the ground a little *east* of the vertical passing through the point from which it starts. This is because the earth's revolution on its axis gives to all objects a motion from west to east, the velocity being greater in proportion to their distance from the axis. Hence the easterly velocity at the summit of a tower is greater than at its base, and a body starting from that point, while falling downwards, must at the same time move toward the east more rapidly than the base of the tower, and hence strikes the ground a slight distance to the east of the vertical. The same explanation applies to the fall of a body down the shaft of a deep mine. A marked deviation of this kind has been shown by experiments made in some of the mines of Saxony. In the case of circus-riders who leap through hoops from the back of a horse running at full speed, and again alight on the animal, it is simply necessary to leap upward and not forward, because their motion in common with the horse carries them onward through the air with the same rapidity as if no leap had been made. Finally, a cannon-ball exerts the same mechanical effect, whether it is fired towards the east, west, north or south, though its velocity with regard to any fixed point in space is vastly different in these different cases, owing to the rotation of the earth. Thus suppose the ball to be propelled from the cannon with a velocity of 470 m. per second. At the equator the rotary motion of the surface of the earth is also 470 m.

per second; hence, if at any station situated in that great circle the ball be fired towards the east, its velocity relatively to the centre of the earth is 470 — 470 m., or 0 m. That is, in the latter case the ball comes to rest relatively to the earth's centre, while the surface moves on, leaving it behind. To us travelling east with the surface, the ball appears to have a westerly velocity of 470 m. per second. The destructive effect of the ball would manifestly be the same in either case.

60. Law III. *Action and reaction are equal and in opposite directions, or the actions of bodies on one another are equal and opposite.*

This law is established solely by induction. To exemplify its truth it will be sufficient to examine a number of phenomena of different classes, and note its applicability to all.

61. Illustrations. A magnet exerts a force of attraction upon a bit of iron, but the iron at the same time attracts the magnet with an equal force. If the magnet is fixed, and the iron free to move, the latter will approach the former. If the iron is fixed and the magnet free, the magnet will move towards the iron. If both are free they will approach each other. And this action is equal as well as mutual and opposite. For it is found by experiment that in the last case the velocity imparted to the iron is as much greater than that imparted to the magnet as its mass is less. That is, if M, M' are the masses, and v, v' the velocities, we shall have $M : M' :: v' : v$. Hence the momenta are equal, for $Mv = M'v'$, and therefore the forces generating these momenta are also equal. The *action* of the magnet, then, is equal to the *reaction* of the iron.

It is to be observed that the terms action and reaction do not denote two forces, but are simply convenient terms by which we express the mutual and opposite actions of the same force. The magnet and iron are drawn together by the single force of their mutual attraction. So when the elasticity of an uncoiling spring pushes apart two bodies connected with it, there is really but one force at work, which moves them in opposite directions, though it may sometimes be convenient to consider the motion as caused by two equal and opposite forces acting from the middle towards the end of the spring. If the masses of the bodies are in any given ratio, as 1 : 2, for example, the velocity impressed on the smaller will be twice that of the latter, so that the momenta will be equal; thus again verifying the law under consideration.

In the case of a bullet fired from a gun, the force of the exploding powder propels the ball forward at a higher velocity. At the same time, however, it forces the gun itself backward, causing the recoil or *kicking* of the piece. If the force of the recoil be measured, it will be found that the momentum of the gun equals that of the bullet. When a man-of-war fires a broadside, the whole vessel sways in the opposite direction. Many other examples may be mentioned. In the exercise of rowing, the water is pushed backward, while the boat moves forward. A person leaping from a small boat pushes it away from him, as he springs away from it. It is because of this principle that a person cannot lift himself into the air by pulling at his boot-straps. The upward pull on the straps (action) is just balanced by the downward push (reaction) of the feet. A case is related of a gentleman who undertook to propel a boat by erecting a large bellows at the stern, and blowing against the sails,[1] the attempt resulting in a signal failure, because the reaction of the current of air as it issued from the bellows neutralized its action upon the sail.

It follows from the law of the equality of action and reaction, that if a body in motion strikes upon one at rest, the shock is the same for both, because the loss of momemtum which the moving body sustains affects it in the same manner as if being at rest it was acted upon by an equivalent force in an opposite direction. If two bodies moving in opposite directions impinge upon one another, the same force is exerted as if one while at rest was struck by the other moving with a momentum equal to their united momenta. This explains why the shock is so violent when two vessels moving in opposite directions run foul of one another. In the case of bodies thus impinging upon each other, though action and reaction are equal, the weaker will evidently be the more injured by the impact. The fist of a pugilist sustains as great a shock as that part of the opponent's body which it strikes, but is not injured, because from its structure it is fitted to endure the shock. But if by accident fist meets fist, one person feels the blow as much as the other.

62. History of the Laws of Motion. Simple and fundamental as are the three laws of motion, it was not until the beginning of the 17th century that they were understood. Kepler, with all his success in astronomy, was ignorant of the principles of inertia. He supposed that moving bodies, if left to themselves, would come to rest, and therefore imagined a constant force acting upon the planets to keep up their velocity. Galileo, during his earlier researches, thought that the only naturally uniform motion was that performed in a circle, but in his *Dialogues on Mechanics*, published in 1638, he gives a correct statement of the first law, though it is not made sufficiently comprehensive in its application. The law, in a general form, was announced by Galileo's pupil, Borelli, in 1667. The truth of that portion of the law relating to the tendency of bodies to move in right lines was recognized by Bendetti, as early as 1585. The principle of the independence of motions was also announced in the *Dialogues on Mechanics*, but its complete demonstration resulted from the establishment of the laws of the motion of the earth by the astronomers of the 17th century, foremost among whom was Sir Isaac Newton. The principles of momentum underlying the third law were known to Galileo, but the laws of impact of bodies, as related to changes in momentum (a knowledge of which was evidently necessary to a general statement of Law III.), were first correctly stated by Wren, Wallis and Huyghens, in papers communicated to the Royal Society about 1609. The terms in which it is now so frequently expressed, "Action and Reaction are equal and opposite," are those used by Newton.

REFERENCES.

For information upon the subject of Units of Force, see
Treatise on Natural Philosophy, by Sir Wm. Thomson and Peter G. Tait; (Oxford, 1867) p. 166.

Numerous examples of the application of the laws of motion will be found in
Elements of Physics, by Neil Arnott.

---

[1] Arnott's *Elements of Physics*.

A Treatise on Mechanics, by Henry Kater and Dionysius Lardner.
Handbooks of Natural Philosophy and Astronomy, by Dionysius Lardner; *First Course*.

For further information upon the subject of a resisting medium see
Outlines of Astronomy, by Sir J. F. W. Herschel; 11th English Ed., §§ 577, 570.
Reports on Observations of Encke's Comet during its Return in 1871, by Asaph Hall and Wm. Harkness (Washington, 1871); p. 33.

CHAPTER V.

COMPOSITION AND RESOLUTION OF MOTIONS AND FORCES.

63. Statics and Dynamics. The science of mechanics is divided into *Statics* and *Dynamics*. Statics treats of balanced forces, or forces in equilibrium; dynamics of the action of forces in producing motion.

The demonstration of the elementary principles of statics requires the preliminary consideration of the composition of motions; that is, of the laws determining the path described by a body in which several motions coexist. The fundamental theorem upon which they all rest is the law of the independence of motions.

64. Parallelogram of Motions. *If a particle be simultaneously impressed with two uniform motions which separately would cause it to describe the adjacent sides of a parallelogram in a given time, it will describe the diagonal of that parallelogram in the same time, and uniformly.*

Suppose a particle at A, Fig. 11, to possess at the same time two uniform motions, one of which would carry it over the line AB, the other AD in a given time. It will move uniformly over the diagonal AC in the same time.

For by Law II. (§ 58, p. 41) each motion takes place as if the other did not exist, hence the motion along AD cannot affect the movement of the particle in a direction parallel to AB, and it will therefore proceed as far in that direction as if it were not impelled along AD; that is, at the end of the given time it must be found somewhere on BC. Also the motion along AD produces its full effect, as if the particle were subject only to it, and hence causes the particle to proceed as far in a direction parallel to AD as if it were not impelled along AB. The particle must therefore be on the line DC at the expiration of the given time; and since it is also on BC it will be found at their intersection C.

This motion is entirely performed in the diagonal *AC*. For let *Ab*, *Ad*, be the distances which would be passed over in any fraction of the whole time if the motions took place separately. Then since, by supposition, the motions over *AB*, *AD* are uniform,

Time of passing over Ab : *Time over AB* : : *Time over Ad* : *Time over AD*; or, as the times are proportional to the spaces described,

$$Ab : AB :: Ad : AD.$$

Hence the parallelograms *Abcd*, *ABCD* are similar, and the point *c* lies in the diagonal *AC*. But, by the first part of the demonstration, c will be the position of the body at the end of the given fraction of time; and as this is true whatever be the time chosen, the whole path must lie in *AC*.

The motion in *AC* is uniform. For, by similarity of triangles, *Ac* : *AC* : : *Ad* : *AD*.

But *Ad* : *AD* : : *Time over Ad* : *Time over AD*,

And *Time over Ad* : *Time over AD* : : *Time over Ac* : *Time over AC*.

Whence *Time over Ac* : *Time over AC* : : *Ac* : *AC*; that is, ***the spaces described are proportional to the times occupied in describing them***, in which case the motion must be uniform.

Also ***the velocity in the diagonal is to the velocity in either side, as the length of the diagonal is to the length of that side***, since each is traversed in the same time.

65. Triangle of Motions. It follows from the preceding proposition that ***if a particle possess simultaneously two uniform motions which, if taking place in succession, and each continuing for the same interval of time, would cause it to describe two sides of a triangle taken in order, it will describe the third side in the same time.*** For the motions which carry a particle over *AD*, *AB*, Fig. 12, if applied at *A*, would by acting successively, carry it over *AD*, *DC*. Hence the effect of their simultaneous occurrence is the same in either case, producing a motion in *AC*, the diagonal of *ADCB*, and third side of the triangle ADC.

66. Component and Resultant Motions. The motions which are thus combined are generally known as *component* or *elementary motions*, while the single motion due to their combination is called the *resultant motion.*

67. Illustrations of Composition of Motion. The composition of motions may be illustrated by placing a ball upon a level square table, and communicating to it two equal impulses along the sides; it will be found to move in the diagonal. If a white ball suspended before a blackboard have a horizontal and a vertical motion communicated to it simultaneously by means of cords, it will be seen to move in an oblique direction. The apparatus shown in Fig. 12 illustrates these principles very clearly. Within a rectangular frame, *ABCD*, slides a second frame, *EFGH*. A white disc *K* slides with an easy motion upon a rod *FN*, attached to the inner frame. A cord attached to the disc runs over a pulley P at the upper extremity of the rod, and is fastened to the outer frame at B. Now it is clear that if the frame $EFGH$ be drawn in the direction of the arrow, the disc K is carried horizontally with the rod FN, while at the same time it moves vertically over FN, owing to the action of the string which is fastened at B. Under the combined effect of these two movements, the disc will be seen to traverse the diagonal GL.

Practically, we notice the composition of motion in the case of a boat rowed across the river, while it is at the same time carried down the stream by the current. The boat moves obliquely, reaching the opposite shore in the same time as if there were no current, but at a point lower down the stream. If the boatman wishes to reach a point directly opposite he must evidently head up the stream, so that the resultant of the combined motions of the boat may lie in a straight line joining the two points. For example, suppose it is wished to cross a stream directly from A to C, Fig. 13, the current being sufficiently strong to carry the boat a distance AD, while the boatman can row it over AB. The boat must be rowed in the line AB, in which case the motions in AB and AD give a resultant motion in AC. The time of crossing is clearly that which it would take to pass from A to B in still water.

68. Polygon of Motions. *If a particle possess simultaneously any number of uniform motions, which occurring successively, and each continuing for the same interval of time, would cause it to describe all the sides but one of a polygon, it will describe the remaining side in the same time, and with a uniform motion.*

Let a body at A, Fig. 14, be impressed with uniform motions, which, if taking place in succession, would cause it to move over the sides AB, BC, CD of the polygon $ABCD$, the time of describing each side being the same. Then by the theorem of the *triangle of motions*, the combination of the movements parallel to AB, BC, gives a resultant motion over AC. Now combine this resultant with the remaining elementary motion parallel to CD. The resultant of these, which is also the resultant of all three motions, is over the line AD. But AD is the fourth side of a quadrilateral, of which AB, BC, CD, are the other three sides. If a fourth elementary motion were present it could be combined with AD, the resultant of the first three (in which case the resultant would be the fifth side of a pentagon), and so on, indefinitely. Hence the proposition is general.

The uniformity of the resultant motion is proved precisely as in the case of two combined movements.

69. Kinematics. The science which treats of the relative motions of bodies considered independently of the causes producing them, is called *Kinematics*.

70. Composition of Forces.

By the aid of the preceding propositions we shall be able to demonstrate the laws of the effect of several forces when acting simultaneously.

Just as elementary or component motions combine to produce a single resultant motion, *component forces* combine to produce a single *resultant force*. The fundamental theorem relative to the composition of forces is the following.

71. Composition of two Forces. — Parallelogram of Forces. *If two forces are represented in magnitude and direction by the adjacent sides of a parallelogram, their resultant will be represented in magnitude and direction by the diagonal.*

Let F, F', be two forces represented in magnitude and direction by the sides AB, AD, of the parallelogram $ABCD$, Fig. 15. Then will their resultant R be represented in magnitude and direction by the diagonal AC. For suppose F, F' to act separately upon a particle at A to produce motion in it. Since forces are proportional to the accelerations which they produce in the same mass, or equal masses (p. 33, Eq. 3), the particle will traverse AB under the sole action of F, in the same time that it will move over AD under the action of F'. The velocities generated by the forces would then be such as to cause the particle to describe two adjacent sides, AB, AD, of a parallelogram in equal times. Hence if both forces act simultaneously, the combination of these velocities will produce a resultant motion represented by AC, which measures the force producing it. But this is evidently R, the resultant force due to the combined action of F and F'. Hence F, F', and R must bear to each other the same relations in magnitude as AB, AD and AC.

AC also represents the *direction* of the resultant because motion can take place only in the line of the force producing it.

72. Triangle of Forces. Since $AB = DC$, and AD, DC, AC, form three sides of a triangle, it is evident that the same forces which are represented in magnitude and direction by AD, AB, AC, two adjacent sides of a parallelogram and its diagonal, are also represented by AD, DC, AC, three sides of a triangle ADC. *Hence, if two forces are represented in magnitude and direction by two sides of a triangle, the third side will represent their resultant.*

73. Particular Cases of Combination of Forces. It is evident that if the angle $A = 180°$, the forces F, F', act in direct opposititition, and the resultant will equal the difference of the components, that is, $R = F - F'$ (9). On the other hand, if $A = 0$, they act together, and the resultant is the sum of the components, that is, $R = F + F'$ (10).[1] Hence if we combine these propositions, and distinguish all forces acting from left to right by the sign $+$, and those acting from right to left by the sign $-$, *the resultant of any number of forces acting in the same straight line equals the algebraic sum of the components.* If R be this resultant, and ΣF the sum of the components,[2] $R = \Sigma F$ (11).

---

[1] Geometrically, this follows from the *Parallelogram of Forces*, because if the angle DAB (Fig. 15) $= 180°$, AC becomes equal to $AB - AD$; while if $DAB = 0°$, AC becomes equal to $AB + AD$.

[2] The Greek letter Σ is frequently used to designate the sum of any finite number of quantities expressed by a symbol placed after it.

74. Illustrations of the Composition of two Forces. The composition of two forces may be illustrated by the case of an arrow fired from a bow. The bowstring ACB, Fig. 16, is brought into a state of tension by the bent bow, hence each half of the string exerts a pull at C towards the extremity of the bow to which it is attached, so that the forces acting upon the arrow lie in the lines CE, CG. Representing their magnitude by CE, CG, the diagonal CF represents the magnitude and direction of the resultant. Hence the arrow, when the string is released, moves forward in the line CF under a force bearing the same relation to the tension of the string in either direction that CF bears to CE or CG.

75. Solution of Problems. Problems relating to the composition of forces are readily solved by the principle of the *Parallelogram or Triangle of Forces*, either by graphical construction, or by the application of trigonometry. Let F and F' be the given components, α the angle made by their lines of action, and R the required resultant. (1.) *Graphical Solution.* — To solve the problem graphically, lay off $AB = F$ units of length, and $AD = F'$ units (Fig. 17), making $BAD = \alpha$. Complete the parallelogram $ABCD$; and the diagonal AC will represent the resultant R; for it is the diagonal of a parallelogram whose sides represent the magnitude and direction of the components. Or lay off $AB = F$ units, $BC = F'$ units, making $ABC = 180° - \alpha$, the *supplement* of the angle made by the directions of the forces. The third side AC of the triangle thus formed, ABC represents R, since AB, BC, represent the components F, F'. (2.) *Trigonometrical Solution.*—Construct the triangle ABC as before. In it are given two sides, AB, BC, and the included angle $ABC = 180° - \alpha$, to find the third side AC.

76. Illustrations. To illustrate these methods let us take a very simple problem. Suppose a canal boat (Fig. 18) to be drawn by two horses, one on each bank, pulling in the directions AB, AC, by means of ropes attached to the bow of the boat, and making equal angles with the line of its keel. Let the pull exerted by each horse be 50 kgrs., the angle of inclination of their lines of action being $BAC = 90°$. It is required to find the resultant effect in moving the boat directly forward. To solve the problem graphically, lay off $AD = 50$ units, $AE = 50$ units. The angle $DAE = 90°$. Complete the parallelogram $ADFE$, and measure the number of units in the diagonal AF, which will be the number of pounds acting to move the boat along AG. The trigonometrical solution requires the determination of the hypothenuse AF of a right-angled triangle AEF, of which two sides, AE and $EF = AD$ are given. As each of these lines $= 50$, $AF = \sqrt{(50^2 + 50^2)} = 70 \cdot 71$ units, whence $R = 70 \cdot 71$ kgrs., or a little more than seven-tenths the sum of the components.

77. Composition of any number of Forces. Polygon of Forces.

If any number of forces are represented in magnitude and direction by all the sides but one of a polygon, the remaining side will represent a single equivalent force.

This proposition follows from the *Parallelogram of Forces* in the same manner as the *Polygon of Motions* follows from the *Parallelogram of Motions*. Let AB, AE, AF, Fig. 19, represent three forces acting at a point A. The resultant of two of them, AB, AE, found by completing the parallelogram $ABCE$, is represented by the diagonal AC. Combining this resultant with the other force *AF*, by completing the parallelogram *ACDF*, we find *AD* to represent the magnitude and direction of the resultant of all three forces. But $AE = BC$, hence the components are also represented by *AB*, *BC*, *CD*, and these lines form three sides of a quadrilateral, of which *AD* is the fourth side. As this process of combination can be pursued indefinitely for any number of additional forces, the proposition is general.

78. Solution of Problems relative to the Composition of more than two Forces. Hence if a number of forces act simultaneously upon a body, their resultant may be found graphically, as follows: Construct a polygon by drawing lines proportional to the forces taken in order and parallel to their directions, and join the extremity of the line representing the last force with the starting point. The last side of the polygon thus formed will represent the resultant required.

For example, suppose a material point *A*, Fig. 20, to be acted upon by four forces, $AB = 5$, $AC = 10$, $AD = 15$. $AE = 20$, so situated that $BAC = 25°$, $CAD = 40°$, $DAE = 75°$. To find their resultant draw $ab = 5$ units of length, and parallel to *AB*; through *b* draw $bc = 10$, and parallel to *AC*; next draw $cd = 15$, and parallel to *AD*, and then $de = 20$, and parallel to *AE*. Finally join *a* and *e*; the length of line *ae* represents the magnitude of resultant, and the angle *bae* is the angle which its direction makes with the direction of the force *AB*.

The value of *ae* can also be determined by trigonometry by calculating successively *ac* from *ab*, *bc* and *abc*, *ad* from *ac*, *ad* and *acd*, and finally *ae* from *ad*, *de* and *ade*.

A simpler method of solution is furnished by analysis, which will be explained in the chapter treating of the analytical method of studying forces.

79. Parallelopiped of Forces. *If three forces not in the same plane are represented in magnitude and direction by three edges of a parallelopiped, the diagonal of that solid will represent a single equivalent force.*

At the point *A* suppose three forces applied, represented by the edges *AE*, *AB*, *AD*, of the parallelopiped *AG*, Fig. 21. Then will the diagonal *AG* represent their resultant. For by the *Parallelogram of Forces* the resultant of *AB* and *AD* is represented by *AC*. But *AC* is one side of a parallelogram *AEGC*, of which *AE*, adjacent to *AC*, is also a side. Hence the resultant of *AC* and the remaining force *AE*, which is the resultant of all three forces, is represented by *AG*, the diagonal of the parallelopiped.

The lines *AB*, $BC = AD$, $CG = AE$, form three sides of a *gauche* or twisted polygon (that is, a polygon whose sides are not all in the same plane), of which *AG* is the fourth side. Hence the present proposition is a particular case of the *Polygon of Forces*, and as the method of combination used could evidently be applied to additional forces lying in different planes, the *Polygon of Forces* is true, whether the lines of action of the forces lie in the same, or in different planes.

80. Equilibrium of Forces. Forces are said to be balanced, or in equilibrium, when their resultant is equal to zero, so that they counteract one another.

81. Equilibrium of two Forces. If two forces are applied at a single point, it is evident that *they will be in equilibrium only when equal and directly opposed.*

82. Equilibrium of three Forces. In the case of three forces in equilibrium, *the resultant of any two of them must be equal and opposite to the third.* For if the resultant of two of the forces is not equal and opposite to the remaining one, it may be combined with the latter, producing an unbalanced resultant force (§ 70), which is contrary to the supposition.

83. *If three forces acting upon a point can be represented in magnitude and direction by three sides of a triangle taken in order, they will be in equilibrium.*

At the point P, Fig. 22, let three forces, F, F', R, be applied in the directions PA, PB, PC, these forces being represented in magnitude and direction by the sides PA, AD, DP, of the triangle PAD *taken in order.* The resultant effect of PA and PB is a force represented by PD, which being the third side of the triangle PAD is equal and opposite PC. Hence the three forces are in equilibrium.

It is to be noted that the forces are represented by the sides *taken in order*; that is, the directions of the forces are those assumed by the sides in going round the triangle. The three forces at P act in the directions of lines drawn from P towards A, from A towards D, and from D towards P.

84. Conversely, *if three forces are in equilibrium about a point, they can be represented in magnitude and direction by three sides of a triangle drawn parallel to their lines of action.*

If the forces PA, PB, PC, are in equilibrium, any one of them, as PC, must be equal and opposite to the resultant PD of the other two. But PA, $AD = PB$, and PD, are three sides of the triangle PAD drawn parallel to the directions of the forces, hence PA, PB and PC, are represented by the three sides taken in order.

85. Since two triangles whose sides are perpendicular, each to each, are similar, it follows that *three forces in equilibrium about a point can be represented by three sides of a triangle drawn at right-angles to their lines of action.*

86. The theorem of the equilibrium of three forces may be put in still another form, which is sometimes useful in practice. From the demonstration of § 82, putting the result in algebraic form, we have,

$$F : R :: PA : PD :: \sin PDA = \sin BPD : \sin PAD. \quad (12.)$$

$$F' : R :: PB : PD :: \sin PDB = \sin APD : \sin PBD. \quad (13.)$$

Also $BPD = 180° - BPC$, $APD = 180° - APC$, $PAD = 180° - APB$, $PBD = 180° - APB$.

Substituting these values in the preceding equations, we have,

$$F : R :: \sin BPC : \sin APB. \quad (14.)$$

$$F' : R :: \sin APC : \sin APB. \quad (15.)$$

Hence, *when three forces are in equilibrium, any one of them is proportional to the sine of the angle included between the directions of the other two.*

87. Equilibrium of any number of Forces. That any number of forces may be in equilibrium about a point, *the resultant of all but one of them must evidently be equal and opposite to the remaining force.*

88. *If the forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium.*

For by a course of reasoning similar to that used in § 82, it will be seen that the resultant of all but one of the forces will be represented by the last side of the polygon, and hence will be equal and opposite to the remaining force.

89. Conversely, *if the forces acting at a point are in equilibrium, they can be represented in magnitude and direction by the sides of a polygon drawn parallel to their lines of action.*

The proof of this proposition is evidently precisely similar to that used in § 83.

90. Experimental Verification of Laws of Equilibrium. The laws of equilibrium of three forces may be shown experimentally by the apparatus represented in Fig. 23. Over two small pulleys, *A*, *B*, a cord is passed, to the ends of which weights are fastened. Weights, *C*, are also suspended from any point *D* on the loop, which is allowed to move freely, and assumes such a position that the three forces acting upon it are in equilibrium. If now we construct a triangle having its sides parallel to *DA*, *DB*, *DC*, respectively, we shall find that the lengths of those sides are proportional to the forces acting at *D*, parallel to their respective directions.

The triangle may be laid off upon paper, or on a blackboard placed behind the apparatus, or better still, we can make use of proportions (14), (15), measuring the angles *ADC*, *BDC*, *ADB*. Each of the three forces will be found to be proportional to the sine of the angle included between the direction of the other two. The angles are most conveniently measured by attaching a small bead to each of the threads (as shown in the figure), a decimetre distant from *D*. Then with a millimetre scale the distances of each bead from the others is measured. These distances will be the chords subtending the angles between the threads, which can then be obtained from a table of chords.[1]

If additional cords with weights attached to their extremities be fastened to *D* and passed over suitably-arranged pulleys, the apparatus of Fig. 25 becomes suitable for verifying the laws of the combination of a greater number of forces. It will be sufficient to allow D to assume its position of equilibrium, and to construct a polygon, having its sides parallel to the directions of the various forces. These sides will be found to have the same relative magnitude as the forces themselves. In the case of a number of combined forces, however, the friction of the pulleys and resistance of the cords causes a considerable variation between the theoretical and experimental results.

---

[1] For a description of several new pieces of apparatus for class experiments relating to this subject, see a paper entitled *Apparatus Illustrating Mechanical Principles*, by R. H. Thurston; published in the *Journal of the Franklin Institute*, 3d Series, Vol. LXII., No. 3, p. 192. Also consult *Reference Table* of works on General Physics.

91. Practical Illustrations. The *Jib and Tie-Rod*, Fig. 24, employed in the common hoisting crane, furnishes a practical illustration of the preceding propositions. In the construction of such a machine, the frame must be made sufficiently strong to sustain the heaviest weight that is to be lifted, and in order to ascertain the strength to be given to each part, the maximum force to which it will be subjected must be known. This can be done by a simple application of the laws of equilibrium. Let P be the maximum downward pressure in kilogrammes, exerted upon the axle of the pulley at S (which can be determined when the maximum weight to be raised, and the angle WSD are known). This causes a certain pull (*strain*) upon the tie AB, in the direction BA, and a compression (*stress*) in the jib AC, in the direction AC. The rod AB will then be stretched, and the jib AC compressed, until the resistances to further change (which act in the directions AO, AM) produce a resultant equal and opposite to P. At this moment let the strain on AB be denoted by T, the stress in AC by T'. P, T and T' are in equilibrium about A, and hence may be represented by three sides of a triangle drawn parallel to their respective directions. From A draw AM, in the line of action of T', MN in that of T, and AN in the direction of P, that is, vertically downward. The forces P, T, T' are proportional to the three sides NA, MN, AM, of the triangle AMN, drawn parallel to their lines of action. That is,

$$P : T :: AN : MN :: BC : BA. \quad (16.)$$

$$P : T' :: AN : AM :: BC : CA. \quad (17.)$$

$$\text{whence } T = P\frac{BA}{BC}. \quad (18.) \qquad T' = P\frac{CA}{BC}. \quad (19.)$$

As P, BA, CA and BC are known quantities, the values of T, T' (expressed in kilogrammes) also become known, and these determined, the proper size of the beams AB, AC, can be calculated by means of the formulæ for the strength of beams.

Another interesting application of the foregoing principles is the mechanical contrivance known as the *Toggle-Joint*, or *Knee-Joint*. Fig. 25. It consists of two bars, AB, AC, connected by a joint at A, the other extremities resting upon the firm base PQ, and the movable plate MN, upon which a powerful pressure is to be exerted. A force P is applied at A by pulling or pushing in the direction AP. This evidently tends to raise MN, and thus presses upon any object confined between that plate and a fixed platform above it. The arms AB, AC, will evidently be compressed until the reactions thus developed are in equilibrium with P. To estimate their magnitude, upon AP lay off AE proportional to P. The reactions T, T', developed in AB, AC, will be represented by EF, FA, as AE, EF, FA, are three sides of a triangle drawn parallel to the directions of the forces in equilibrium. Hence

$$P : T :: AE : EF :: \sin AFE : \sin FAE. \quad (20.)$$

$$P : T' :: AE : FA :: \sin AFE : \sin FEA. \quad (21.)$$

$$\text{whence } T = P\frac{\sin FAE}{\sin AFE}. \quad (22.) \qquad T' = P\frac{\sin FEA}{\sin AFE}. \quad (23.)$$

An inspection of the figure will show that as MN rises under the influence of the force P, the angle FAD becomes more obtuse. whence also FAE and $DAE = FEA$ also increase, while AFE diminishes. Hence, under these circumstances the values of T, T', as given in (22), (23), also increase, becoming greater and greater as FAD approaches 180°, when T and T' equal infinity.[1] By the action of a comparatively small force we may thus produce an enormous pressure. The Toggle-Joint is frequently used in printing presses for bringing the type and paper into close contact, in machines for cutting large thicknesses of paper, in the cotton-presses at Mobile, etc. Its great advantage is that when in operation the pressure exerted by it increases simultaneously with the increase of resistance caused by the compression of the substance acted upon; it has the disadvantage, however, that to obtain a large range of vertical movement of the plate MN, the dimensions of the press must be very considerable.

92. The equilibrium of forces when the point of application is at rest, is known as *statical equilibrium*. When the forces acting upon a body in motion are balanced, we have *dynamical equilibrium*, the treatment of which is reserved for a future portion of this work.

93. Resolution of Forces.

In paragraphs 70–78 we have explained the manner in which several forces can be replaced by a single resultant. We now take up the converse problem of finding two or more component forces, which are equivalent to a single force. This process is known as the *resolution of forces.*

Suppose AC, Fig 26, represent any force, and let it be required to find two other forces which acting together would produce the same effect as AC. It is merely necessary to construct any parallelogram $ABCD$ on AC as a diagonal. The sides AB, AD, will represent the required components. Or the triangle ADC may be constructed, in which case AD, DC, represent the components.

Since any number of triangles, ABC, ADC, AEC, AFC, Fig. 27, can be constructed on a single line taken as a base, it follows that if AC represent a force the lines AB and BC, AD and DC, AE and EC, AF and FC, will equally represent two component forces. Hence *any force can be resolved in an indefinite number of ways.*

To resolve a force into two components *whose directions are given,* we must know the angles BAC, BCA, between those directions and AC. We have then given two angles and the included side of a triangle, to find the remaining sides, which represent the components sought. This may be done either by trigonometry or graphically.

94. Resolution of a Force into two rectangular Components.

It is frequently necessary to resolve a force into two components at right angles to each other. This is done by constructing a rectangle, or a right-angled triangle, upon a line representing the given force. Thus if AC, Fig. 28, represent this force, AB and AD, or DC and AD, will represent the components.

---

[1] Since in this case FAE and FEA each $= 90°$, and $AFE = 0°$, and $T = T' = P\frac{\sin 90°}{\sin 0°} = P\frac{1}{0} = \infty$.

Calling R the original force, F the component represented by AB or DC, F' that represented by AD, and denoting the angle $BAC = ACD$ by α, in which case $DAC = 90° - \alpha$, we have $AB = AC \cos \alpha$, $AD = AC \sin \alpha$, whence $F = R \cos \alpha$ (24), $F' = R \sin \alpha$ (25), general equations for the resolution of a force into two rectangular components.

95. Equation of Relation between Components and Resultant. Denote the forces represented by AC, AB, BC, Fig. 27, by R, F, F', respectively, and call $BAC = \alpha$. $BCA = \beta$. Then as $AC = AB \cos BAM + BC \cos BCM$, $R = F \cos \alpha + F' \cos \beta$. (26.) Hence *the resultant of any two forces is equal to the algebraic sum of the products of each component into the cosine of the angle which it makes with the resultant.*

96. Examples of Resolution of Forces. Let PQ, Fig. 29, be a body of weight W resting upon a horizontal table MN. It is required to find the downward pressure upon the table when PQ is acted upon by a force R, represented by CA. Let R be resolved into two rectangular components, one F, represented by CD, at right angles to MN, the other, F', represented by RB, parallel to MN. Of these forces, F alone exerts pressure upon the table, F' merely tending to make the body PQ slide along the surface on which it rests. Hence the whole downward pressure $= W + F$; or as $F = R \sin \alpha$, downward pressure $= W + R \sin \alpha$. The tendency to move horizontally is evidently $R \cos \alpha$.

Or, as another example, suppose that a body weighing W kilogrammes is to be raised by means of ropes AE, AF, Fig. 30, and it is wished to determine the number of kilogrammes which must be exerted at the end of each rope, in order to start it. Draw the vertical AC, having a length of W units, and complete the parallelogram $ABCD$ by drawing BC, CD, parallel to AF, AE. Resolve W into two components, one T, parallel to AE, and a second T' parallel to AF. Then T, T' will be the forces which must be applied at E and F to lift W. From the triangle of forces ABC we have

$$W : T :: AC : AB, \text{ whence } T = W\frac{AB}{AC} = W\frac{\sin DAC}{\sin CBA}. \quad (27.)$$

$$\text{and} \quad W : T' :: AC : BC, \text{ whence } T' = W\frac{BC}{AC} = W\frac{\sin BAC}{\sin CBA}. \quad (28.)$$

from which T, T' becomes known if the angles made by each rope with the vertical are determined.

The examples given on p. 53 to explain the subject of statical equilibrium, also serve equally well to illustrate the resolution of forces.

97. Resolution of a Force into any number of Components. A force can also be directly resolved into any number of components by the principle of the *Polygon of Forces*, these components being in one or several planes. It is frequently simpler, however, to resolve the original force into two components, then one of these components into two others, and so on.

98. Practical Examples. As a good example of this, let us take the case of a vessel propelled by the wind. AB, Fig. 31, is a boat which is moved forward by the action of the wind blowing in the direction indicated by the arrow *W*. *MN* is the sail. To show how the vessel proceeds under an oblique wind, let *OP* represent the magnitude of the whole pressure of the wind upon the sail *MN*. Resolve it into two components, *OD* perpendicular to *MN*, and *OC* parallel to it. The parallel component *OC* can have no effect to move the vessel because it acts upon the sail edge-ways; hence the whole moving force is that due to the perpendicular component *OD*. Resolve *OD* into two other components, *OE* and *OF*, one of which, *OE*, is parallel to the keel of the vessel, and the other, *OF*, perpendicular to it. Then *OE* alone acts to push the vessel directly forwards, while *OF* tends to push it sideways. Hence the boat moves forward with a certain velocity due to the component *OE*, while at the same time it has a slight motion sideways (*leeway*), caused by the component *OF*. The reason why there is so little movement sideways in proportion to that forwards, is because of the greater resistance to motion in the former direction, caused by the shape of the vessel.

The manner in which two vessels can sail in different, and even opposite directions with the same wind, is easily demonstrated by the laws of the resolution of forces. In Fig. 32 the vessel *AB* is represented as proceeding in a direction opposite to that in which it moved in Fig. 31. The direction of the wind is the same as before, but the sail *MN* is placed in a new position. Resolving *OP* into *GD* and *OC* as before, *OC* is inoperative; *OD* can be again resolved into *OE* parallel to the keel, and *OF* perpendicular to it. Hence the vessel moves on in the line *BA* under the action of the component *OE*.

By varying the position of the sail, a vessel can be made to proceed in various directions, while the wind remains unchanged. In a boat with but one sail, the greatest advantage is gained when the wind is parallel to the keel, as its whole force is then exerted in causing a motion forward. The vessel is then said to be *scudding* or *sailing before the wind*. A ship sailing against the wind as closely as possible is said to be *close-hauled*. A large ship can sail so that her keel makes an angle of but six points (67° 30′) with the wind, and smaller vessels can sail very much closer. In a vessel with a large number of sails, a very favorable position of the wind is when it is at right angles to the keel (*upon the beam*), as the sails are then all acted upon with equal force, while if the wind is parallel to the keel the aft sails cut it off from those in front of them

---

CHAPTER VI.

REACTION OF SURFACES. — COMPOSITION AND RESOLUTION OF FORCES ACTING AT DIFFERENT POINTS OF A BODY.

99. Equilibrium sustained by Reaction. When a body acted upon by a system of forces is at rest, equilibrium is often sustained by the reaction of one or more surfaces with which the body is in contact. The simplest example of this is when a body rests upon a horizontal plane, in which case the reaction caused by the compression of the material of which the plane is composed, is equal to the weight of the body. Additional examples will be found in the case of the *Jib and Tie-Rod* and the *Toggle Joint* already described.

100. Position of Resultant. If one surface be pressed against another in any manner, the resultant of the reactions at all the different points of its surface must be equal and opposite to the resultant of all the forces acting upon it.

101. Body pressed against a Curved Surface. When a body is pressed against a curved surface by the action of any number of forces, if there is equilibrium, the resultant of these must be normal to that surface. For if this were not the case, the resultant might be resolved into two other forces at right angles to each other, one normal to the curved surface, and the other tangential to it. The former would be opposed by the reaction of the surface, while the latter, being unopposed by any force, would produce motion over it, which is contrary to the supposition. The resultant must also pass through the point of contact of the surface, as otherwise there would be a tendency to rotate about that point. Thus let B, Fig. 33, be a cube pressed against the sphere A. For equilibrium, the resultant of the forces acting on A must pass through the point of contact P, and be perpendicular to the curved surface of A.

102. Constrained Bodies. If a body is in such a condition that motion can take place only in certain directions, it is said to be *constrained*. Thus a body fastened by a pivot is constrained to turn about that pivot. A body fastened at two points is constrained to move about an axis joining these two points, and if fastened at three points not in the same straight line, it is capable of no motion whatever.

103. Action of Forces on Constrained Bodies. It will be profitable to examine a few cases of the action of forces on constrained bodies.

The simplest case is that of a body resting upon an inclined plane. Let A, Fig. 34, be such a body, and let its weight, which acts vertically downward, be represented by the line W. Resolving this into two components, P perpendicular to the plane L, and F parallel to it, it is evident that the whole force of the component P is exerted in producing a pressure upon the plane at right angles to its surface, while the other component F tends to produce motion along L. A similar case of constrained motion will be noticed in the case of a body resting upon a horizontal surface, and acted upon by an oblique force.[1]

As another example, let P be a ring hung upon a curved wire AB, Fig. 35. If it be acted upon by a force F, normal to the curve, it will remain at rest, being kept in equilibrium by the reaction $-F$, equal and opposite to F. If, however, the force lie in any other direction, as in the line PR, it can be resolved into two components, one of which along PF is normal to the curve, and the other along PF' tangential to it. The former of these will be balanced by the reaction of the wire, while the latter component will cause the ring to move along the wire.

---

[1] See p. 55.

When a body is fastened upon a pivot, any force F, Fig. 36, whose direction is in the line joining its point of application S with the pivot P, is directly opposed by the reaction of P. If the force has any other direction, as R, it will cause revolution about the pivot until the line PSR becomes straight, the component F producing pressure on P, while F' causes the body to rotate. F' evidently diminishes as the angle PSR increases, until it equals 0 when that angle assumes a value of 180°.

104. Forces applied at different Points of a Body. Hitherto we have treated of forces applied at a single point. We now proceed to consider their action when applied at different points of any connected system of particles. The fundamental principle upon which our reasoning is based is the *Transferability of Force.*

The simplest case of transference is that of a force acting upon a rod in the direction of its length. The rod may be considered as composed of a line of particles, *abcd*, etc., Fig. 37. Now if a pull be exerted at A, the particle a is moved from its normal position, so that its distance from b is increased. This develops a molecular attractive force which acts upon b, causing it to move slightly towards a; thus in its turn c is acted upon, and so on through the whole length of the rod, until the molecular tension exerted between each particle is the same throughout the rod, and equal to the force applied at A. The rod is then in a state of equilibrium, and the particle n exerts a pull upon any point, as B, to which it is attached, equal to the force acting at A. Hence this power appears to be *transferred* from A to B. If, on the other hand, the rod is pushed at A in the direction AB, the distance between a and b is lessened, and the force thus developed is excited from particle to particle until the whole rod is in a state of tension. The particle n then exerts a force upon B equal to the pressure applied at A.

It follows from what precedes, that the effect upon B will be the same at whatever point of AB the power is applied. Hence *equal and opposite forces applied at the ends of a rod or rope, are in equilibrium.* For each may be supposed to be applied to any single particle taken in their line of action. Also the resultant of two or more forces thus applied must equal their algebraic sum.

105. Principle of Transferability of Force. *The efficiency of any force acting upon a body is not altered by transferring its point of application to any point in its line of action.* Thus let F be a force acting upon a body AB, Fig. 38. Evidently the effect of F is the same, whether it be applied at a, b, c, d or e.

This can be illustrated experimentally by balancing AB on a pivot P. The force F being applied at a, let it be counterpoised by a weight W, suspended from B. If then the point of application of F be changed from a to b, c, d or e, the weight will still be found to balance it exactly.

106. Composition and Resolution of Forces applied at different Points. The principle of transference furnishes a ready method of determining the resultant effect of two or more forces acting at different points of a body. Let F, F', be two forces applied to the body AB, Fig. 39, at the points M, N, and acting along the lines FM, $F'N$. Suppose these lines to be produced till they meet in some point S, which may lie either within or without the body AB. Then since S is in the continuation of FM, the effect of F will be the same as if applied at that point, and hence may be transferred to it. Also since S is in the continuation of $F'N$, F' may also be transferred in the same manner. The combined effect of the two forces is therefore the same as if both were applied at S, the intersection of their lines of action. In this case the resultant would be found, as already explained, by means of the parallelogram, so that if Sa, Sb represent F and F', SC will represent R, their resultant. Hence, to find the effect of two forces applied at different points of a body, *prolong their lines of action until they meet in a point, and proceed to find the resultant as if both forces acted at that point.*

The resultant of any number of forces applied at different points of a body may be found by combining them two by two.

107. Equilibrium of Forces applied at different Points. The resultant of F and F' would be balanced by the application of an equal and opposite force anywhere in the line SR. Hence, *if a body is kept at rest by the action of three oblique forces applied at different points, these forces would be in equilibrium if applied at a single point.*

108. Effect of Pivot. If a pivot be placed anywhere in the line of action of the resultant SR, as at C, the resultant may be considered as applied directly at that point, in which case it merely exerts a pressure upon the pivot without producing any tendency in the body to rotate around it. If the pivot does not lie in SR there will evidently be a tendency to rotation. Hence in the case of a body resting upon a pivot, *if the resultant of all the applied forces passes through the pivot, there will be equilibrium.* Conversely, *if the applied forces are in equilibrium the resultant passes through the pivot.*

CHAPTER VII.

Statical Moments.—Parallel Forces.—Couples.

Statical Moments.

109. Moment of a Force. *The tendency of a force to rotate a body about a fixed point is measured by the product of its intensity into the perpendicular distance from the point to the line of action of the force.* This product is called the *moment* of the force.

Thus the moment of the force F, Fig. 40, relatively to C is $F \times Cm$. Let F, F', Fig. 40, be two forces applied at M, N, and tending to cause rotation in opposite directions about a pivot placed at the point C, and let R be their resultant. When R passes through C there will be equilibrium among the forces F, F', and the reaction of the pivot, hence the tendencies of F, F' to rotate AB will in that case be equal. From C draw Cm, Cn, respectively, perpendicular to the directions of F, F', and construct a parallelogram $SbCa$, the adjacent sides of which, Sa, Sb, represent the magnitude and direction of F, F'. Then

$$F : F' :: Sa : Sb :: \sin bSC : \sin aSC :: \frac{Cn}{Sc} : \frac{Cm}{Sc} : Cn : Cm,$$

$$\text{whence } F \times Cm = F' \times Cn. \quad (27.)$$

But $F \times Cm$ is the moment of F, and $F' \times Cn$ the moment of F', relatively to C, and since we have shown that when their moments are equal the forces are balanced about C, these moments must represent the efficiency of the forces F, F', to cause rotation in either direction about that point.

Hence, calling F any force, l the perpendicular distance from any point to its line of action, and M its moment relatively to that point, we have

$$M = Fl. \quad (28.)$$

The perpendicular l is called the *arm* of the moment.

The direction of the tendency to rotation is said to be *right-handed* when it is in the direction of the moment of the hands of a watch, *i. e.*, from left to right, and *left-handed*, when in an opposite direction. Thus in Fig. 40, F tends to cause a right-handed, and F' a left-handed rotation. The former are generally designated by the sign $+$, the latter by the sign $-$.

110. Experimental Verification. The laws of moments may be verified experimentally by means of the apparatus represented in Fig. 41. AB is a disc of wood balanced on a pivot passing through C, so as to remain at rest indifferently in any position. A weight D is then attached at B by a cord, while a second cord is fastened to any other point of AB as K, and passed over a pulley M. Weights E are then attached to the latter cord, and the body AB is allowed to assume its position of equilibrium. There is then equilibrium between the moments of E and D. The perpendiculars CB, CL, dropped from C upon the directions of the forces, are then measured, and it will be found that $E \times CL = D \times BC$, in whatever position on the disc the point of attachment K be taken.

111. Resultant Moment of several Forces. If two or more forces tend to produce rotation in the same direction their united efficiency (*resultant moment*) is evidently equal to the sum of their moments.[1] If they tend to produce rotation in opposite directions, their united efficiency is equal to the excess of the sum of the moments in one direction over the sum of those in the opposite direction. Hence calling right-handed rotations +, and left-handed rotations —, *the resultant moment of any number of forces relatively to a point is equal to the algebraic sum of the moments of the components.* Or, calling the resultant moment M_r,

$$M_r = \Sigma Fl. \quad (29.)$$

The moment of the resultant of two forces acting to produce similar rotations is equal to the sum of the moments of the components, since the resultant may be substituted for them without change of efficiency; and the moment of the resultant of two forces acting to produce dissimilar rotations is equal to the difference of the moments of the components, for a like reason. Hence as this course of reasoning may be extended to any number of forces, it follows that *the moment of the resultant of any combination of forces equals the algebraic sum of the moments of the components.* Or, calling ΣFl this sum, R the resultant of all the forces, and l_0 the length of its arm,

$$Rl_0 = \Sigma Fl. \quad (30.)$$

As the preceding propositions hold, whatever may be the inclination of the forces to each other, they are true when the forces are parallel.

112. Equilibrium of Moments. When the sum of the moments in one direction equals the sum of the opposite moments, the resultant moment = 0, which is expressed algebraically,

$$Rl_0 = \Sigma Fl = 0. \quad (31.)$$

To produce this, R may become zero, or the arm l_0 may assume that value. If the latter is the case, there is a simple tendency to a translatory movement of the body along the line of action of the resultant. And if the body be fastened at any point in that line, the force acting upon it will be balanced by the reaction of the pivot or other support on which it rests, as already shown.

113. Moment of a Force relatively to an Axis. The moment of a force relatively to an axis is its tendency to produce rotation about that axis. Let R be a force represented by PR, Fig. 42, and AB an axis. Draw PM perpendicular to both PR and AB. If now the force PR be resolved into two components, one, F, represented by PF, perpendicular to AB, and the other F', represented by PF', parallel to AB, the former alone will have a tendency to produce rotation about that axis. The moment of the whole force R relatively to AB is therefore the same as the moment of its component F, which is evidently the moment of F relatively to the point M, that is to $F \times PM$.

---

[1] In this chapter we consider only the case in which all the forces lie in the same plane.

Hence, *the moment of a force relatively to an axis is equal to the perpendicular distance between the axis and the line of action of the force, into that one of the components which is at right angles to the axis.*

Forces are in equilibrium about an axis when the algebraic sum of their moments relatively to it is zero.

114. Practical Application. An interesting application of the principle of equilibrium of forces about an axis is the following. If three equal weights, P, P, P, Fig. 43, be applied at equal distances from each other and from an axis through O, the algebraic sum of their moments relatively to that axis will always be equal to zero, whatever may be the absolute position of the points of application, A, B, C. The algebraic sum of the moments of P, P, P, is $P \times OH \times P + OL - P \times OD = P\ (OH + OL - OD)$, and it is to be proved that this product $= 0$. Join B, C, bisect BC in G and draw AG. As ABC is an equilateral triangle AG is perpendicular to BC and passes through O. Also $AO = 2OG$, whence $OD = 2OK$. Now $OH + OL = OH + OK + KL = 2OK$, as $HL = KL$ because of similarity of the triangles BKL, CKH. Hence $OH + OL - OD = OD - OD = 0$, and $P\ (OH + OL - OD) = 0$ for any position of A, B, C.

This is practically applied in pumping-machines in which three pumps are worked by a single shaft, their piston-rods being attached to three cranks, making angles of 120° with each other. The resistance of the three pumps is the same for all positions of the cranks, and perfect steadiness of motion is gained.

Parallel Forces.

115. Resultant of two Parallel Forces. *The resultant of two parallel forces acting in the same direction is parallel to them, and equal to their sum.*

Let F, F_1', Fig. 44, be any two forces applied at the points A, B. Their resultant, R, is found by prolonging FA, $F_1'B$, till they meet in P, and constructing a parallelogram of forces, $bPac$. It is clear that R may be considered as applied at P, and that its direction will lie between FA and $F_1'B$. Now imagine $F_1'B$ to be revolved about B in the direction indicated by the arrow. The point P continually recedes until the line $F_1'B$ assumes the position F_2B, parallel to FA, when P is at an infinite distance, and in this case, as all lines meeting at an infinite distance are parallel, PR, the direction of the resultant, must then be parallel to FA and $F_2'B$.

Also, for all inclinations of the components to each other, the magnitude of the resultant R of the forces F, F' is given by the equation $R = F \cos bPc + F' \cos aPc$ (26, p. 55). But when $F'B$ becomes parallel to FA, $bPc = 0$, $aPc = 0$, whence $R = F + F'$. (32.)

116. Point of Application of Resultant. To find the point of application of the resultant on the line AB, suppose M, Fig. 46, to be that point. Imagine a force R' equal and opposite to the resultant R to be applied there. This force would balance the resultant, and must therefore be in equilibrium with the components F, F'. Hence the moments of F, F' relatively to M, must be equal, that is, $F \times Ma = F' \times Mb$, whence $F : F' :: Mb : Ma$. But by similarity of triangles, $Mb : Ma :: MB : MA$, whence $F : F' :: MB : MA$. (33.) That is, *the resultant divides the line joining the points of application of the component forces into parts whose lengths are inversely proportional to the adjacent components.*

117. Two Parallel Forces in Opposite Directions. *If the forces are in opposite directions, the resultant is parallel to them, and equal to their difference. It lies on the same side of both, next the greater component, and has the same direction as that component.*

The three forces, F, F', R', Fig. 45, are in equilibrium, hence the resultant of F and R' must be equal and opposite to F'; that is, equal to $R' - F$ (Eq. 32). It is also evident from the figure that the resultant is on the same side of both F and R', next R', and in the same direction with it.

The point of application, B, is so situated that the moments of F, R' are equal, relatively to it, in which case $F : R' :: MB : AB$. (34.)

118. Practical Illustration. To illustrate the application of the preceding propositions, suppose two weights of 10 and 30 kgrs. respectively to be hung upon the extremities of a rod AB, Fig. 45, 2 metres in length. It is required to find the force R' which is requisite to support them, and the point at which it must be applied, neglecting the weight of the rod. We have $R' = R = \mathrm{F} + \mathrm{F}'$, and as $F = 10$ kgrs, $F' = 30$ kgrs., $F' = 40$ kgrs. Also $F : F' :: MB : MA$, or denoting AB by l and MB by x, $F : F' :: x$ $l - x$ whence $x = l \frac{\mathrm{F}}{F + F'}$ (35) Substituting for l, F, F', their values as given above $x = 2 \times \frac{10}{40} = \frac{1}{2}$ m., and $l - x = 1\frac{1}{2}$ m. That is, the force R' must be applied at a distance of $\frac{1}{2}$ m. from M.

119. Resolution of a Force into two parallel Components. Proportions (33) (34) furnish a method of resolving a single force into two parallel forces applied at given points. Thus, let it be required to resolve the force R into two parallel forces applied at points A, B. In the proportion $F : F' :: MB : MA$ we know MB, MA, and in the equation $R' = R = F + F'$, we know R. From these the value of F, F' can readily be computed. If F, F' are to have opposite directions, proportion (34), and the corresponding equation would be used in the same manner. It is also evident that if the magnitude of the components is given, their points of application can readily be found by the same proportions.

To illustrate, if the force $R' = 40$ kgrs. is to be resolved into two parallel components F, F' having the same direction, and applied at points A, B, distant from M, $1\frac{1}{2}$ m. and $\frac{1}{2}$ m. respectively, the magnitude of these is found as follows: $R = F + F' = 40$ kgr. $F : F' :: MB : MA :: \frac{1}{2} : 1\frac{1}{2}$ whence $F' = 3F$, and $R = F + F' = 4F = 40$ kgrs., whence $F = 10$ kgrs., $F' = 30$ kgrs.

120. Centre of Parallel Forces. Since the point of application M of the resultant of the parallel forces, F, F' divides the line AB joining the points of application of the forces in a fixed ratio depending only upon the magnitude of F, F', it follows that whatever may be the direction of the forces F, F', R, so long as their relative intensities remain unchanged, the position of M does not vary. This point is called the *Centre of the Parallel Forces.*

121. Resultant of any number of Parallel Forces. The position and magnitude of the resultant of more than two parallel forces may be found by combining them successively. Thus let F, F', F'', Fig. 46, be three parallel forces applied to a body at the points A, B, C. To find their resultant we first combine F and F'. Joining A and B we have from (33)

$$F : F' :: MB : MA,$$

which determines the point M. Also, $F = F + F'$. We next combine this resultant R with the remaining component F''. Joining MC, we have $R = F + F' : F'' :: CN : MN$, which determines the point of application N of the resultant R' of all three forces. Also $R' = R + F'' = F + F' + F''$. R' is evidently parallel to the components. The position of the centre, N, can be found either by a graphical construction, or calculated by the rules of trigonometry when the positions of A, B, C, are known.

If any one of the forces, as F'', acted in an opposite direction, we should evidently have $R' = F + F' - F''$, and its position would be found by (34).

As this process of combination can be continued indefinitely, it follows that calling forces acting in one direction +, those in an opposite direction —, *the resultant of any number of parallel forces equals their algebraic sum*, that is,

$$R = \Sigma F. \quad (36.)$$

The preceding method of determining the position of the point of application of the resultant is simple when there are but few forces to be considered, but becomes very tedious with numerous forces. Another method is then adopted, which will be explained in the following chapter.

122. Experimental Demonstration. The laws of parallel forces can be demonstrated experimentally by means of a very simple apparatus represented in Fig. 47. A graduated bar, AB, is suspended from the hooks of two delicate spring balances D, E. The weight of AB causes a certain constant depression of the indexes of the balances, which is read once for all. A weight W is then placed anywhere upon AB, as at C, and the balance readings taken anew, which being diminished by the preceding readings, show the pressures exerted by it at G and H, the points of suspension. Calling these F, F', it will be found that in whatever position W may be placed, $W = F + F'$, and $F : F' :: CH : CG$. For verifying the laws with a greater number of parallel forces additional balances may be placed between D and E.

Couples.

123. Definition. If two equal and opposite parallel forces not acting in the same straight line be applied to a body, their algebraic sum is 0, and hence they have no tendency to produce a motion of translation. But as their resultant moment with regard to any point can never equal 0, their effect will be to rotate the body about an axis.

Such a combination of forces is called a *couple*. The perpendicular distance between the lines of action of the forces is called the *arm of the couple*, and the plane containing these lines of action the *plane of the couple*. Any line at right angles to this plane is an *axis of the couple*. The terms *right-handed* and *left-handed* are applied to couples in the same manner as in the case of moments.

124. Efficiency of a Couple. *The rotary effect, or moment of a couple is measured by the product of either force into the length of the arm.*

Let F, F', Fig. 48, constitute a couple whose arm is AB. Assume an axis at right angles to the plane of F, F', passing through any point P. The moment of F relatively to $P = F \times PB$, and the moment of F' relatively to the same point $= F' \times PA = F \times PA$, as $F' = F$. The resultant moment of the two forces is therefore equal to

$$F \times PB + F \times PA = F(PB + PA) = F \times AB.$$

If the axis be taken at a point not between the lines of action of F and F', as P', we have *moment of force F relatively to $P' = F \times P'B$, moment of force F relatively to $P' = -F' \times P'A = -F \times P'A$*, as the moment is left-handed. Hence, *resultant moment of F and $F' = F \times P'B - F \times P'A = F(P'B - P'A) = F \times AB$*, as before.

Hence calling M the moment of the couple, F the magnitude of the force acting at each end, and l the arm, $M = Fl$. (37.)

125. Transference of Couples. *A couple may be turned in its own plane, or moved parallel to itself without altering its efficiency.*

For the effect of the couple F, F', Fig. 49, to cause rotation about P, is evidently not altered if the forces assume the positions f, f', as the arm MN equals AB. With any axis, as P', not situated between the lines of action of F, F', we have *Rotary-effect* $= f \times mn = f \times MN = F \times AB$. The moving of the couple to any other position in the same plane would simply be equivalent to moving the assumed axis to some other point, as from P to P', which does not alter the efficiency.

A couple may also be transferred to any plane parallel to its own plane without alteration of its efficiency, because its whole tendency being to produce rotation about an axis at right angles to its plane, the effect will be the same in whatever plane at right angles to the axis it may lie.

126. Reduction of Couples. From Eq. (37) it follows that couples are equivalent when their moments are equal. That is, a couple of force $= 8$ and arm $= 3$, has the same efficiency as one of force $= 2$ and arm $= 12$, as in either case $M = Fl = 24$. Hence *any couple may be replaced by an equivalent one having a given arm.* The value of the force in this case is found from the equation $M = Fl$, whence $F = \frac{M}{l}$ (38). If, on the other hand, we wish the new couple to have a given force, we may find the corresponding length of the arm from the equation $l = \frac{M}{F}$ (39). This process is called the reduction of couples to the same arm, or the same forces. For example, let it be required to replace the couple $F = 5$, $l = 10$, by an equivalent one having an arm $= 2$. From (38) we find the corresponding value of F to be $\frac{M}{l} = \frac{50}{2} = 25$. An equivalent couple having an arm 2 must then have a force of 25.

127. Equilibrium of Couples. Since the sole effect of a couple is to produce rotation, no single force can be so applied as to hold it in equilibrium. It will evidently be balanced, however, by the application of an equal couple acting in an opposite direction.

128. Combination of Couples having the same Axis. *The resultant moment of any number of couples lying in the same or parallel planes, is equal to the algebraic sum of their separate moments.*

For they may all be reduced to equal arms, and so applied that the forces act in the same straight line. Calling Fl, $F'l$, $F''l$, $-F'''l$, the reduced couples, these will be equivalent to a single couple with a force $= F + F' + F'' - F'''$, and an arm l, whose efficiency equals the algebraic sum of the component couples.

Hence denoting the resultant moment of any number of couples by M_r, we have $M_r = \Sigma Fl$. (40.)

For equilibrium we must have $\Sigma Fl = 0$. (41.)

129. Representation of Couples by Lines. In many cases, especially where the planes of the couples are inclined to each other, it is convenient to represent the direction, magnitude and intensity of a couple in the following manner. Let F, F', Fig. 48, be the couple which is to be represented. From any point O draw a line OM at right angles to the plane in such a direction that to an observer looking from O towards M the couple shall seem right-handed, and let the length of OM be made proportional to the number of units in the moment of the couple.

130. Combination of Couples having different Axes. Couples lying in planes inclined to each other can be combined by means of a theorem similar to the *Parallelogram of Forces*. Let F, F', f, f', Fig. 50, be two couples reduced to the same arm AB and acting in planes $FAF'\ faf'$, making with each other any angle FAf. Let the lines FA, fA, $F'B$, $f'B$ represent the magnitudes of the forces of the couple acting at the extremities of AB. The two forces F, f, give a resultant R represented in magnitude and direction by AR; and $F'\ f'$ give an equal and parallel resultant R', represented by BR'. The four forces F, f, $F'\ f'$ may therefore be replaced by two parallel forces R, R', lying in the plane RAB. The resultant moment evidently equals $R \times AB$. Hence couples may be combined and resolved in the same manner as ordinary forces. If instead of the forces F, $F'\ f$, f' we represent by AF, AF', Af, Af' the *moments* of the couples, AR, AR', will evidently represent the moment of the resultant couple.

Since the plane of a couple and its axis of rotation are at right angles the axes have the same inclinations to each other as the planes. Hence the moments of F, f may be laid off on the axes AF, Af, Fig. 51, instead of in the planes BC, BE. The line AR will then represent the moment of the resultant couple, the plane of which is BD, at right angles with AR. We have, therefore, the general proposition, *if the moments and axes of two couples be represented by two sides of a parallelogram, the diagonal of that parallelogram represents the moment and axis of the resultant couple.*

131. Illustrations. The effect of two equal and opposite forces to cause rotation is seen in spinning a common humming-top, the pull exerted on the string and applied at the circumference of the axle being one of the forces, and the reaction of the handle applied at the axis of symmetry of the axle the other. Another excellent example is the case of a light sphere of cork or wood kept rapidly rotating in the air by the action of a fountain-jet playing against the under side of it. The upward force of the jet applied on the surface is equal and opposite to the force of gravity (or weight of the ball) applied at the centre of the sphere, thus forming a couple.

CHAPTER VIII.

ANALYTICAL STATICS.

132. General Principles, Definitions, etc. When we have a large number of forces to consider, the processes of composition and resolution of forces are very much simplified by the use of analytical methods.

It is customary in this case to refer the directions of the forces, their points of application, etc., to coördinate axes at right angles to each other, as in Analytical Geometry.

To the student unacquainted with that study the following explanation will give the main principles required for an intelligent study of the present chapter.

When the forces are all in the same plane we make use of two rectangular axes OX, OY, Fig. 52, known as the *axis of X* and *axis of Y* respectively, and collectively as the *axes of coördinates*. Lines drawn from any point parallel to the axis of Y and terminated by the axis of X are called *ordinates*, and are generally designated by the letter y, while lines drawn from a point parallel to the axis of X and terminated by the axis of Y are called *abscissas*, and are designated by the letter x. The abscissas and ordinates of a point taken together are called the *coördinates* of that point. Thus BM and $BN = MO$ are coördinates of B, BM being the ordinate, BN the abscissa. The point O at the intersection of the axes is the *origin of coördinates*. Abscissas of points lying to the right of the axis of Y are generally affected by the sign $+$; of those lying to the left, $-$. Ordinates of points lying above the axis of X are $+$; of points below, $-$. Evidently the values of x and y for any point fix its position; thus the position of B is known when we have given $OM = x$, $MB = y$. To illustrate the application of this to mechanics; two forces of magnitude 5 and 4 applied at a single point, and making angles of 45° and 30° with any given line are represented by taking that line as the axis of X (or Y if preferable), and the point of application as O, and laying off two lines OA, OB of lengths $= 5$ and 4 units respectively, making angles of 45° and 30° with OX.

When the forces do not lie in a single plane they can be referred to three rectangular axes, X, Y, Z, Fig. 54, formed by the intersection of three planes at right angles to each other. The position of a force in space is then represented by the angles α, β, γ, which its direction makes with the axes of X, Y and Z. The coördinates of any point referred to a system of three coördinate axes are denoted by the letters x, y, z, indicating distances parallel to X, Y or Z respectively.

The *projection* of a line upon a plane is the distance between perpendiculars drawn to the plane from each of its extremities. The projection of a line upon another line is the distance between perpendiculars drawn to the latter line from the extremities of the former. The projection of a line upon a plane or a second line is equal to the length of the line into the cosine of angle which it makes with the plane or line upon which it is projected.

133. Analytical Statics in Two Dimensions.—Forces applied at a single Point. Let F, represented by OB, Fig. 52, be any force applied at a single point. It can be resolved into two rectangular components represented by MO and BM. Denote the angle BOM by α.

Then, as $MO = BO \cos \alpha$, $BM = BO \sin \alpha$, it is clear that the component parallel to $X = F \cos \alpha$, and that parallel to $Y = F \sin \alpha$.

The direction of the components will be found by observing the algebraic signs of $\cos \alpha$ and $\sin \alpha$. Thus if the force F lie in the direction indicated by the line OC, the angle α which it makes with X is MOC (counted on the same side of OX with OB), and both $\cos \alpha$ and $\sin \alpha$ are *minus*, MOC being greater than 180°. In this case the components would be $-F \cos \alpha$, parallel to X, $-F \sin \alpha$ parallel to Y.

134. Resultant of any Number of Forces. Suppose that any number of forces. F, F', F'', F''', F^n, are applied at a single point, and it is required to find their resultant. Take the point of application as the origin of coördinates, and denote by α, α', α'', α''', α^n, the angles made by the directions of the forces with the axis of X. Denote the resultant by R, and call θ the angle which it makes with X. Resolving F, F', etc., into their components parallel to X and to Y, as explained in the preceding paragraph, and taking care to observe the algebraic signs of the functions $\sin \alpha$, $\cos \alpha$, etc., the results of the following tables are obtained.

Components parallel to X.	*Components parallel to Y.*
$F \cos \alpha$	$F \sin \alpha$
$F' \cos \alpha'$	$F' \sin \alpha'$
$F'' \cos \alpha''$	$F'' \sin \alpha''$
$F''' \cos \alpha'''$	$F''' \sin \alpha'''$
$F^n \cos \alpha^n$	$F^n \sin \alpha^n$

Hence, *Total force along*

$$X = F \cos \alpha + F' \cos \alpha' + F'' \cos \alpha'' + F''' \cos \alpha''' + F^n \cos \alpha^n = \Sigma F \cos \alpha.$$

Total force along

$$Y = F \sin \alpha + F' \sin \alpha' + F'' \sin \alpha'' + F''' \sin \alpha''' + F^n \sin \alpha^n = \Sigma F \sin \alpha.$$

Now conceive the resultant R to be resolved in the same manner. Its components are

$$R \cos \theta, \textit{ along } X, \qquad R \sin \theta, \textit{ along } Y.$$

But the components of R along X must evidently be equal to the sum of the components of the elementary forces F, F', etc., along the same axis, and the components of R along Y must be equal to the sum of the components of F, F', etc., along that axis. Hence

$$R \cos \theta = \Sigma F \cos \alpha. \quad (42.) \qquad R \sin \theta = \Sigma F \sin \alpha. \quad (43.)$$

Squaring these equations and adding the results,

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = (\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2, \text{ or as } \cos^2 \alpha + \sin^2 \alpha = 1,$$

$$R^2 = (\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2; \text{ whence}$$

$$R = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2}. \quad (44.)$$

Equation (44) gives the magnitude of the resultant. It remains only to determine its direction by means of the angle θ.

Dividing Eq. (43) by Eq. (42), $\frac{R \sin \alpha}{R \cos \alpha} = \frac{\Sigma F \sin \alpha}{\Sigma F \cos \alpha}$, whence

$$\tan \theta = \frac{\Sigma F \sin \alpha}{\Sigma F \cos \alpha}. \quad (45.)$$

The algebraic sign of tang θ evidently follows from the signs of $\Sigma F \sin \alpha$, $\Sigma F \cos \alpha$.

The value of R may also be obtained (having first determined the angle θ) from the equation $R \cos \theta = \Sigma F \cos \alpha$, whence

$$R = \frac{\Sigma F \cos \alpha}{\cos \theta} = (\Sigma F \cos \alpha) \sec \theta. \quad (46.)$$

That there may be equilibrium among a system of forces R must $= 0$, whence $(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2 = 0$, (47), which equation can only be satisfied when each of its terms $= 0$. Hence the equations of equilibrium are

$$(\Sigma F \cos \alpha) = 0, \quad (48.)$$
$$(\Sigma F \sin \alpha) = 0. \quad (49.)$$

135. Resultant of Oblique Forces applied at several Points. In this case there is a tendency to rotation about an axis produced as well as a resultant translatory force.

Let F, F', F'', F^n be forces applied at points whose coördinates relative to an assumed set of coördinate axes X, Y, Fig. 53, are (x, y), (x', y'), (x'', y''), (x^n, y^n). It is required to find the magnitude and direction of the resultant, and also the resultant couples produced. Their directions are denoted by the angles α, α', α'', α^n, as before.

Let us first consider the case of a single force F applied at the point S (x, y).

Resolving it into two components we have

$$P = F \cos \alpha = \textit{component parallel to } X.$$
$$Q = F \sin \alpha = \textit{component parallel to } Y.$$

Now let us imagine two opposite forces K, $-K$, each equal and parallel to P, to be applied at O. Also two opposite forces T, $-T$, each equal and parallel to Q. The action of the system of forces F, F', etc., will not be altered by this addition. We have now acting upon the body, (1) at the point S (x, y), the forces $P = F \cos \alpha$, parallel to X, and $Q = F \sin \alpha$ parallel to Y; (2) at the point O, the forces $+K$, $-K$, along X, and $+T$, $-T$, along Y.

The combined effect of these forces is as follows; — the forces K and T are together equal to a single force

$$L = \sqrt{K^2 + T^2} = \sqrt{(F \cos \alpha)^2 + (F \sin \alpha)^2} = F,$$

making an angle β with X of such value that

$$\text{tang } \beta = \frac{T}{K} = \frac{Q}{P} = \frac{F \sin \alpha}{F \cos \alpha} = \text{tang } \alpha, \text{ whence } \beta = \alpha,$$

that is, L is equal and parallel to F and applied at O.

The equal and opposite forces P and $-K$ form a right-handed couple, having an arm $SM = y$, and whose moment is consequently $Py = yF \cos \alpha$. Also the equal and opposite forces Q and $-T$ form a left-handed couple with an arm $SN = x$, and whose moment is $-Qx = -xF \sin \alpha$. These couples being in the same plane, form a resultant couple

$$M = Py - Qx = yF \cos \alpha - xF \sin \alpha.$$

Now suppose this process to be repeated for each of the forces F', F'', F^n. We shall then have a similar result for each: that is, the force F' will be replaced by an equal and parallel force applied at O, and a couple whose moment $= y'F' \cos \alpha' - x'F' \sin \alpha'$; the force F^n will be replaced by an equal and parallel force applied at O, and a couple $y^nF^n \cos \alpha^n - x^nF^n \sin \alpha^n$, and so for all the forces. Their combined efficiency will therefore be the resultant of F, F', F'', F^n applied at O, and the resultant of all the couples, which has a moment equal to the algebraic sum of the elementary couples. Using the same notation for the resultant as in the preceding cases, we have as the joint effect of all the forces,

$$R = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \sin \alpha)^2}, \text{ applied at } O, \quad (50.)$$

$$\text{tang } \theta = \frac{\Sigma F \cos \alpha}{\Sigma F \sin \alpha}, \quad (51.)$$

and the couple of moment $= \Sigma\,(yF \cos \alpha - x\,F \sin \alpha)$. (52.)

If $\Sigma\,(yF \cos \alpha - xF \sin \alpha) = 0$, the resultant passes through O. If $R = 0$ and $\Sigma\,(yF \cos \alpha - xF \sin \alpha)$ does not $= 0$, the resultant is simply a couple.

When there is equilibrium among the forces applied at different points of a body

$$R = 0, \text{ whence } (\Sigma F \cos \alpha) = 0,\ (\Sigma F \sin \alpha) = 0, \quad (53.)$$
$$\text{and } \Sigma\,(yF \cos \alpha - xF \sin \alpha) = 0. \quad (54.)$$

136. Analytical Statics in three Dimensions. — Resolution of a Single Force into three Rectangular Components. Let F, Fig. 54, be any force in space which is required to be resolved into three components at right angles to each other. Take the point of application of F as the origin of coördinates, and let OF represent that force, its direction being indicated by the values of α, β, γ, the angles which its line of action makes with the axes X, Y, Z respectively. Construct the rectangular parallelopiped $OBFE$ having OF as its diagonal. The adjacent edges OA, OB, OC will represent the components required (§ 79, p. 50). It only remains to find analytical expressions for these in terms of F.

Project OF upon the axes OX, OY, OZ, successively. The figures $ODFA$, $OEFB$, $OCFG$, all being rectangles,

$$OA = OF \cos \alpha,\ OB = OF \cos \beta,\ OC = OF \cos \gamma.$$

But OA, OB, OC represent the rectangular components of F, hence denoting these by P, Q, S, we have

$$P = F \cos \alpha \ (53). \quad Q = F \cos \beta \ (54). \quad S = F \cos \gamma \ (55.)$$

137. Composition of Three Forces at Right Angles. Conversely, the resultant of three rectangular components can be found analytically as follows: —

Squaring equations (53), (54), (55), and adding the resultant we have

$$P^2 + Q^2 + S^2 = F^2 \cos^2 \alpha + F^2 \cos^2 \beta + F^2 \cos^2 \gamma =$$
$$F^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

But as $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ (by a proposition of analytical geometry), $F^2 = P^2 + Q^2 + R^2$, and $F = \sqrt{P^2 + Q^2 + R^2}$. (56.)

The same result may be reached geometrically, as

$$OF^2 = OA^2 + OB^2 + OC^2. \quad (57.)$$

138. Resultant of any Number of Forces in Space applied at a Single Point. Let F, F', F'', F^n be any forces applied at a single point. Take their point of application as the origin of coördinates, and denote the angles made by the forces, and the axes of coördinates by the letters α, β, γ, α', β', γ', α'', β'', γ'', α^n, β^n, γ^n respectively. Denote their resultant by R, and let its direction be indicated by the angles α^r, β^r, γ^r, which it makes with X, Y, Z.

Now resolve F, F', etc., into rectangular components along X, Y, Z, as explained in § 136, taking care to observe the algebraic signs of the cosines of the direction-angles. These components are

Along X.	*Along Y.*	*Along Z.*
$F \cos \alpha$	$F \cos \beta$	$F \cos \gamma$
$F' \cos \alpha'$	$F' \cos \beta'$	$F' \cos \gamma'$
$F'' \cos \alpha''$	$F'' \cos \beta''$	$F'' \cos \gamma''$
$F^n \cos \alpha^n$	$F^n \cos \beta^n$	$F^n \cos \gamma^n$

From these it follows that

Total force along $X =$
$$F \cos \alpha + F' \cos \alpha' + F'' \cos \alpha'' + F^n \cos \alpha^n = \Sigma F \cos \alpha.$$

Total force along $Y =$
$$F \cos \beta + F' \cos \beta' + F'' \cos \beta'' + F^n \cos \beta^n = \Sigma F \cos \beta.$$

Total force along $Z =$
$$F \cos \gamma + F' \cos \gamma' + F'' \cos \gamma'' + F^n \cos \gamma^n = \Sigma F \cos \gamma.$$

Now conceive the resultant R to be resolved into three components along the axes of coördinates. These will be

$$R \cos \alpha^r \text{ along } X.$$
$$R \cos \beta^r \text{ along } Y.$$
$$R \cos \gamma^r \text{ along } Z.$$

But the rectangular components of R along X, Y, Z respectively must evidently be equal to the sum of the components F, F', F'', F^n along the same axes. Hence

$$R \cos \alpha^r = \Sigma F \cos \alpha, \quad (58.)$$
$$R \cos \beta^r = \Sigma F \cos \beta, \quad (59\ )$$
$$R \cos \gamma^r = \Sigma F \cos \gamma. \quad (60.)$$

Squaring (58), (59), (60) and adding the results,

$$R^2 (\cos^2 \alpha^r + \cos^2 \beta^r + \cos^2 \gamma^r = (\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2,$$

$$\text{or } R = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2}. \quad (61.)$$

This same value of R may also be found by considering R as a single force with components $\Sigma F \cos \alpha$, $\Sigma F \cos \beta$, $\Sigma F \cos \gamma$, and using the same geometrical reasoning as in the case of determining any force from its rectangular components as already explained (§ 137).

The direction of R is given by the equations

$$\cos \alpha^r = \frac{\Sigma F \cos \alpha}{R}, \ (62.) \quad \cos \beta^r = \frac{\Sigma F \cos \beta}{R}, \ (63.) \quad \cos \gamma^r = \frac{\Sigma F \cos \gamma}{R} \ (64.)$$

For equilibrium,

$$R = \sqrt{(\Sigma F \cos \alpha)^2 + (\Sigma F \cos \beta)^2 + (\Sigma F \cos \gamma)^2} = 0, \quad (65.)$$

in which case

$$(\Sigma F \cos \alpha)^2 = O, \quad (\Sigma F \cos \beta)^2 = O, \quad (\Sigma F \cos \gamma)^2 = 0. \quad (66.)$$

Hence equations (66) are the equations of equilibrium.

When forces in space are applied at different points they tend to produce a translatory motion, and also rotation about each of the three axes X, Y, Z. This case being quite complex, the student is referred for its demonstration to more extended works on mechanics.

139. Resultant and Centre of any Number of Parallel Forces in Space. A much simpler method of obtaining the position of the centre of parallel forces than the construction explained in § 121, p. 64 is afforded by analytical methods based on the principle of moments.

Let R be a force applied at the point S whose coördinates are x^n, y^n, z^d (Fig. 55). If R be parallel to OY its moment relatively to the axis OZ is Rx. If parallel to OZ its moment relatively to OX is Ry, and if parallel to OX its moment relatively to OY is Rz.

Now suppose any number of parallel forces F, F', F^n applied at points (x, y, z), (x, y', z'), (x^n, y^n, z^n) to act upon a body, and suppose S, Fig. 56, to be the point of application of their resultant R (centre of parallel forces). The magnitude of the resultant is equal to the sum of the components or, $R = \Sigma F$. The coördinates of S are determined as follows: — Since the parallel forces composing the system may be turned in any direction without altering the position of the centre (§ 120, p. 64), we may consider them as acting parallel to each of the axes successively. First let us suppose them to be parallel to OY. The moment of the force F relatively to OZ is then Fx; that of F', $F'x'$; that of F^n, $F^n x^n$. The sum of these moments is ΣFx, and this must be equal to the moment of the resultant, which is $x^r \Sigma F$. Hence $x^r\, \Sigma F = \Sigma Fx$. Next suppose all the forces to be parallel to OZ. By the same course of reasoning we have $y^r\, \Sigma F = \Sigma F\, y$. Then supposing all the forces to be parallel to OX, we have $z^r \Sigma F = \Sigma F z$ From these three equations

$$x^r \Sigma F = \Sigma Fx, \quad (67.)$$
$$y^r \Sigma F = \Sigma Fy, \quad (68.)$$
$$z^r \Sigma F = \Sigma Fz, \quad (69.)$$

we find the required coördinates to be

$$x^r = \frac{\Sigma Fx}{\Sigma F}, \ (70.) \quad y^r = \frac{\Sigma Fy}{\Sigma F}, \ (71.) \quad z^r = \frac{\Sigma Fz}{\Sigma F}, \ (72.)$$

For equilibrium there must be no tendency either to translation or rotation, hence $R = \Sigma F = 0$, (73.) $\Sigma F\,x = 0, \quad \Sigma F\,y = 0, \quad \Sigma F\,z = 0$, (74.) are the equations of equilibrium.

140. Virtual Velocities.— Definition. Let F, Fig. 56, be a force applied at the point A, and suppose this point of application to be moved by some external force through an extremely small space, so that it assumes the position A' or A'', the force F still acting in the same direction as at first. From A' or A'' draw a perpendicular $A'B$ or $A''B'$. The distance AB or AB' intercepted between the original point of application A and the foot of this perpendicular, is called the *virtual velocity* of the force F, and is positive when it is laid off in the direction in which the force acts, and negative when opposite to that direction. Thus AB is positive, AB' negative.

141. Principle of Virtual Velocities. *If the forces acting upon a body are in equilibrium, and a very small displacement be given to the body, the algebraic sum of the products of each force into its virtual velocity is zero.*

If the forces are applied at a single point, represent this point by A, Fig. 57, and the forces themselves by F, F', F'', F^n. Suppose the point of application to be moved over the very small distance AA'. Call $\alpha, \alpha', \alpha'', \alpha^n$ the angles made with X by the directions of the forces, $\beta, \beta', \beta'', \beta^n$ the angles made with Y, and denote the angle between X and the line AA' by θ. AB is the virtual velocity of F, which may be called v. Denote the virtual velocities of the other forces by v', v'', v^n respectively. It is to be proved that $\Sigma Fv = 0$.

For any single force, as F,

$$v = AB = AA' \cos BAA' = AA' \cos (\alpha - \theta) =$$
$$AA' (\cos \alpha \cos \theta + \sin \alpha \sin \theta),$$

$$\text{whence } Fv = F \times AA' (\cos \alpha \cos \theta + \sin \alpha \sin \theta). \quad (75.)$$

In like manner for F', F'', F^n we have

$$F'v' = F' \times AA' (\cos \alpha' \cos \theta + \sin \alpha' \sin \theta), \quad (76.)$$
$$F''v'' = F'' \times AA' (\cos \alpha'' \cos \theta + \sin \alpha'' \sin \theta), \quad (77.)$$
$$F^n v^n = F^n \times AA' (\cos \alpha^n \cos \theta + \sin \alpha^n \sin \theta), \quad (78.)$$

Taking the sum of these equations, noting that $\alpha, \beta, \alpha', \beta'$, etc., are complements, so that $\sin \alpha = \cos \beta$, $\sin \alpha' = \cos \beta'$, etc.,

$$Fv + F'v' + F''v'' + F^nv^n = \Sigma Fv = AA' [\cos \theta\, (F \cos \alpha + F' \cos \alpha' + F'' \cos \alpha'' + F^n \cos \alpha^n) + \sin \theta\, (F \cos \beta + F' \cos \beta' + F'' \cos \beta'' + F^n \cos \beta^n)]. \quad (79.)$$

But, as by supposition, there is equilibrium of the forces about A, it follows from equations (48), (49), p. 70, that

$$\Sigma F \cos \alpha = F \cos \alpha + F' \cos \alpha' + F'' \cos \alpha'' + F^n \cos \alpha^n = 0,$$
$$\Sigma F \cos \beta = F \cos \beta + F' \cos \beta' + F'' \cos \beta'' + F^n \cos \beta^n = 0,.$$

and as the terms of equation (79) enclosed between the brackets become $= 0$, the whole member $= 0$, and hence $\Sigma Fv = 0$. (80.)

It follows from (80) that the resultant of any number of forces multiplied by its virtual velocity equals the algebraic sum of each of its components multiplied into the corresponding virtual velocity, that is, if we call V the virtual velocity of the resultant R, $RV = \Sigma Fv$. (81.)

In case the forces are not applied at a single point, the theorem still holds. For suppose the forces F, F', F'', F^{n} to be applied at different points. It follows from the principle explained in § 106 (p. 59) that any two of them, as F, F', produce the same effect as if they were both applied at the intersection of their lines of action. Imagine them thus applied, call R the resultant equal to their combined effect, and denote by m, n, the virtual velocities of F, F', when transferred to the point supposed. According to the principles stated in the preceding paragraph,

$$RV = Fm + F'n. \quad (82.)$$

Now the virtual velocities of F, F' are not affected by the transference of their points of application. For suppose A, D, Fig. 58. to be these points, and let the body be slightly displaced by rotation, though a very small angle a about an axis passing through any point M, and at right angles to the plane of the forces. The points A, D, move through very small arcs AC, DF, which may be considered as straight lines at right angles to the radii MA, MD. Also $AC = MAa$, $DF = MDa$. AB is the virtual velocity of F, DE of F'. If F, F', were transferred to G, their virtual velocities would be GK, GI. It is to be proved that $GK = AB$. We have

$$v = AB = AC \cos BAC = MAa \cos BAC = MAa \cos AMS =$$
$$MAa\frac{MS}{MA} = MSa$$

And $m = GK = GH \cos HGK = MG^{a} \cos HGK = MGa \cos GMS =$ $MGa \dfrac{MS}{MG} = MSa$; whence $m = v$. In like manner it could be shown that $GI = DE$, or $n = v'$. Hence substituting these values in (82),

$$RV = Fv + F'v'. \quad (83.)$$

Now imagine R to be combined in the same manner with one of the remaining forces F'', call R' the resultant of these, V' its virtual velocity. By the same course of reasoning as that just employed,

$$R'V' = RV + F''v'' = Fv + F'v' + F''v''. \quad (84.)$$

In like manner,

$$R''V'' = R'V' + F^{n}v^{n} = Fv + F'v' + F''v'' + F^{n}v^{n} = \Sigma Fv. \quad (85.)$$

But if there is equilibrium among the forces acting upon the body $R'' = 0$, whence in that case $\Sigma Fv = 0$. (86.) Hence the theorem of virtual velocities is true whether the forces act upon the body at a single point, or have several points of application.

The theorem also holds when the forces are in different planes, but the demonstration of this case is too complicated for the purpose of the present work.

142. Conversely, *if* $\Sigma Fv = 0$, *there will be equilibrium among all the forces.* For from (85) $\Sigma Fv = R''V''$, whence if $\Sigma Fv = 0$, $R'' = 0$. In this case the forces must either balance or form a couple. If they form a couple, the addition of an equal and opposite couple would produce equilibrium. Let P, P', u, u' be the forces and virtual velocities of such an equal and opposite couple. By the proposition $\Sigma Fv + Pu + P'u' = 0$, hence $Pu + P'u' = 0$. As P and P' are parallel forces having different points of application, this equation can hold only when both forces $= 0$.

The theorems of Virtual Velocities are of great practical importance, as they comprehend all the principles of the equilibrium of forces. A valuable application of them will be explained in treating of the mechanical powers (Chapter X.).

CHAPTER IX.

CENTRE OF GRAVITY.

143. Definition. All bodies tend towards the earth by virtue of their weight. Since this tendency is impressed upon every particle, a body is really acted upon by a system of parallel forces having a vertical direction. The resultant of all these must equal their sum, which is evidently the weight of the whole mass. The centre of the system remains at the same point, whatever may be the direction of the forces relatively to the body (§ 120, p. 64), that is, in whatever position it may be placed. This centre is called the *centre of gravity* of the body.

Hence the *centre of gravity* of a body, or system of bodies, may be defined as the point through which the resultant of the weights of the component particles always passes, whatever position the body or system may occupy.

In considering masses as acted upon by gravitation, the whole weight may be supposed to be concentrated at the centre of gravity, so that if this point be supported the body will therefore be balanced about it. Owing to these facts, a knowledge of the location of this point is often of great importance in mechanical problems, and we therefore proceed to a consideration of the methods of determining its position in various cases.

144. Centre of Gravity of Lines.[1] *The centre of gravity of a material straight line is at its middle point*, for the resultant of the weights of the two extreme particles lies at a point midway between them. The same is true for the next two, and thus we may proceed with all the particles.

The centre of gravity of two straight lines inclined at an angle, lies upon the line joining their middle points. Thus let AB, AC, Fig. 59, be two lines united at A. Let G' be the middle point of AB, G'' of AC. The whole weight of AB may be considered as concentrated at G', that of AC at G''. Joining G', G'', the centre of gravity G of the whole system will evidently lie upon the line $G'G''$. To find the position of G we have merely to determine the point of application of the resultant of the weights of AB, AC. Since the moments of these weights about G must be equal,

$$G'G : G''G :: weight\ AC : weight\ AB. \quad (87.)$$

The centre of gravity of a curved line, EBF, Fig. 60, lies somewhere within the segment ABC.

---

[1] In strict language, we should speak of the centre of gravity of a thin rod instead of the centre of gravity of a line, and of a thin sheet instead of a surface, but as the weights of portions of a homogeneous prismatic or cylindrical rod are proportional to their length, and the weights of sheets of uniform thickness to their surface, we may for the sake of brevity make use of the former terms without inconvenience, provided that their true meaning be kept in mind.

145. Centre of Gravity of Surfaces. *The centre of gravity of any plane surface which is symmetrical about a line lies upon that line.* For let $ABCD$, Fig. 60, be any such surface symmetrical about BD. As there is the same quantity of matter on either side of BD, the centre of gravity of the surface must lie upon it. The line BD is called an axis of symmetry.

If any surface has two or more axes of symmetry, its centre of gravity lies at the intersection of those axes. Let $ABCD$ be such a surface, symmetrical about BD and AC. From what we have already said, it is evident that the centre of gravity lies on both BD and AC. Hence it is at G, their intersection.

The centre of gravity of a parallelogram is therefore at the intersection of its diagonals; that of a circle, ellipse, or any polygon, at its centre of figure. The centre of gravity of a ring is at its geometrical centre, and the same is true of the perimeters of all polygons.

146. Centre of Gravity of a Triangle. *The centre of gravity of a triangle is on a line joining the vertex with the middle of the base, at a distance from the vertex equal to two-thirds the length of that line.* Let ABC, Fig. 61, be a triangle. From A draw AD to the middle point D of BC, also from B draw BE to the middle point E of AC. The lines AD, BE, each divide the triangle into two equal parts; hence the centre of gravity must lie on each line, and is therefore at G, their point of intersection. To determine the position of G, draw ED. ED is parallel to AB because it divides AC, BC, proportionally. Hence the triangles ABC, EDC, are similar, and

$$ED : AB :: DC : BC.$$

Also as ABG, EDG are similar,

$$ED : AB :: GD : GA,$$

whence
$$DC : BC :: GD : GA.$$

But $DC = \frac{1}{2}BC$ by construction, hence $GD = \frac{1}{2}GA = \frac{1}{3}AD$, or $AG = \frac{2}{3}AD$.

147. Centre of Gravity of any Polygon. The centre of gravity of any polygon may be found by dividing it into triangles. Thus let $ABCDE$, Fig. 62, be a polygon. Drawing BE, BD, it is divided into three triangles whose centres of gravity are at g, g', g''. Join g, g'. The centre of gravity, g''', of the portion $ABDE$, must lie on gg', and may be found from the proportion $gg''' : g'g''' :: EBD : ABE$, as the weights of EBD, ABE, may be supposed to be concentrated at g, g'. Now join $g'''g''$. The centre of gravity of the polygon $ABCDE$ must lie on this line at a point G, so situated that

$$g'''G : g''G :: ABE + EBD : DBC. \quad (88.)$$

This process can evidently be applied to polygons having any number of sides, and G may be found either graphically or by trigonometrical calculation.

148. Centre of Gravity of Solids. In the case of homogenous solids symmetrical about any plane, the centre of gravity evidently lies in that plane. Hence the planes of symmetry determine the position of the centre of gravity. If a solid has two planes of symmetry its centre of gravity lies in their line of intersection, if three, at their point of intersection. Hence the centre of gravity of a cube, sphere or any regular polyedron is at its centre of figure.

The centre of gravity of lines, surfaces or solids, with the exception of the most simple ones, are best found by analytical methods requiring the use of the Calculus.

149. Centre of Gravity of Pyramid. As it is frequently necessary in physical problems to know the position of the centre of gravity of a pyramid, we give a geometrical demonstration.

The centre of gravity of a pyramid lies in the line joining the vertex with the centre of gravity of the base, at a distance from the vertex equal to three-fourths the length of that line.

a. Triangular Base. Let $ABCD$, Fig. 63, be a pyramid having a triangular base ABC. Draw AE, BF from the vertices A, B, of the triangles ADC, BDC to the middle of their common base CD. Let g be the centre of gravity of BDC, g' that of ADC. Draw Ag, Bg'. The centre of gravity of the pyramid lies in Ag, for the pyramid may be considered as built up of a series of triangular prisms with bases parallel to BDC, and having a very small altitude; the centre of gravity of each of these lies in Ag, hence the centre of gravity of the whole pyramid lies in that line. It also lies in Bg', since B may be taken as the vertex, and ACD as the base of the pyramid, which may then be considered as composed of a series of prisms with bases parallel to ADC, and having infinitesimal altitudes. Hence as Ag, Bg' are in the same plane ABE, it must be at G, their point of intersection. To determine the position of G join gg'. As gGg', AGB are similar triangles,

$$gg' : AB :: Gg : GA.$$

And as gEg', BEA are similar,

$$gg' : AB :: gE : BE.$$

Combining these proportions,

$$gE : BE :: Gg : GA$$

But $gE = \frac{1}{3} BE$, therefore $Gg = \frac{1}{3} GA = \frac{3}{4} Ag$.

b. Any Base. If the base of the pyramid is not a triangle let it be any polygon whatever, $BCDEF$, Fig. 64. Divide the polygon into triangles BCD, BDE, etc., and pass planes ABD, ABE through the vertex A, and the diagonals BD, BE. The whole pyramid is thus divided into a series of triangular pyramids. Let g be the centre of gravity of the polygonal base, and draw Ag. The centre of gravity of the pyramid is upon this line, since that solid may be supposed to be built up of a series of prisms with bases parallel to $BCDEF$, and of very small altitude, each of whose centres of gravity (and hence the centre of gravity of the whole solid) would be on Ag. It also lies in the plane of the centres of gravity of the triangular pyramids, into which the whole pyramid has been decomposed, that is, in a plane parallel to the base $BCDEF$, and cutting Ag at a distance of three-fourths its length from A. Hence it must be at the intersection of that plane with Ag, or at a point G so situated that $AG = \frac{3}{4}\, Ag$.

150. Centre of Gravity of Cone. Since a cone may be considered as a pyramid with an infinite number of faces, it follows that the centre of gravity of a cone is situated on the line joining its vertex with the centre of its base, at a distance from the vertex equal to $\frac{3}{4}$ the length of the line.

151. Centre of Gravity of Systems of Bodies. The centre of gravity of connected systems is readily found by the application of the preceding principles.

Let A, B, Fig. 65, be a system composed of two bodies. The mass of A may be considered as concentrated at its centre of gravity g, that of B at g'. The centre of gravity of the system must evidently lie at some point G on gg', so situated that the moments of A and B relatively to it are equal. Hence its position may be determined from the proportion

$$Gg : Gg' :: \textit{weight } B : \textit{weight } A. \quad (89.)$$

In the case of a system containing more than two bodies the position of the centre of gravity may be determined by a process similar to that used in the case of an irregular polygon (§ 146, p. 76). The application of these principles to extended systems of bodies is of great importance in astronomy.

152. Centre of Gravity of Two Bodies connected by a homogenous Rod. To find the centre of gravity of two bodies connected by a rod we first find the centre of gravity of the bodies themselves, and then find the centre of gravity of the rod. By considering the weight of bodies as concentrated at their centres of gravity, and that of the rod as concentrated at its middle point, we readily determine the centre of gravity of the whole system. Thus let A, B, Fig. 65, be the bodies, and gg' the homogenous rod connecting them. The centre of gravity of A and B is at G, that of gg' at C. The centre of gravity of the whole system is at x, so situated that

$$Cx : Gx :: \textit{weight of } A + B : \textit{weight of } gg'. \quad (90.)$$

153. Distance of Centre of Gravity of two or more Bodies from a Plane. *The distance of the centre of gravity of any number of bodies from a plane is equal to the quotient arising from dividing the product of the weight of each body into the distance of its centre of gravity from that plane, by the sum of the weights of the bodies.*

This follows directly from equations (70), (71), (72), p. 73.

For let ZOY, Fig. 56, be the plane in question. The weights of the bodies may be represented by F, F', F'', F^n, and considered as parallel forces applied at points (x, y, z), (x', y', z'), etc. The distances of the centres of gravity of the separate bodies from ZOY are x, x', x'', x^n. The centre of gravity of the system of bodies will therefore be the centre of these parallel forces, which lies at a distance x_r from ZOY, so situated that $x_r = \frac{\Sigma F x}{\Sigma F}$.

154. General Analytical Methods. The equations of § 136 (70, 71, 72) furnish a general method of determining the position of the centre of gravity of a system of bodies by reference to trilinear coördinates. If we substitute for F, F', etc., in those equations the quantities M, M', etc., the weights of the bodies, and denote by (x, y, z), (x', y', z'), etc., the coördinates of their centres of gravity, the coördinates x_r, y_r, z_r will give the position of the common centre of gravity of the system, this point being the centre of a combination of parallel forces M, M', M'', M^n applied at points (x, y, x), (x', y', z'), etc. Hence we obtain the following equations, in which x_r, y_r, z_r are the coördinates of the centre of gravity of the system of bodies M, M', M'', M^n,

$$x_r = \frac{\Sigma Mx}{\Sigma M} \quad (91.) \qquad y_r = \frac{\Sigma My}{\Sigma M} \quad (92.) \qquad z_r = \frac{\Sigma Mz}{\Sigma M} \quad (93.)$$

155. Application of the Calculus. The processes of integral calculus enable us to apply the preceding method to the determination of the position of the centre of gravity in the case of a body of any regular geometrical form, as all bodies may be considered as systems of particles. Equations (91), (92), (93) are evidently true, whatever the magnitude of the masses composing them; hence, if we imagine a body to be divided into a series of infinitesimal weight-elements, they still hold. Suppose this division to take place in the case of any body, and let dM be the element of weight, x, y, z, the coördinates of its centre of gravity. The moments of any weight-element relatively to the axes Z, X, Y respectively are xdM, ydM, zdM, and the sum of the moments represented in (91), (92), (93), by ΣMx, ΣMy, ΣMz, must be replaced by $\int xdM$, $\int ydM$, $\int zdM$, as dM is an infinitesimal. Also ΣM must be replaced by $\int dM$. Hence for any body whatever the coördinates of the centre of gravity are

$$x_r = \frac{\int xdM}{\int dM}, \quad (94.) \qquad y_r = \frac{\int ydM}{\int dM}, \quad (95.) \qquad z_r = \frac{\int zdM}{\int dM}. \quad (96.)$$

If the body under consideration is homogeneous dM is equal to its volume, which is a volume element dV, multiplied by the weight of a unit of mass. As this latter factor is a constant, for homogeneous bodies

$$x_r = \frac{\int xdV}{\int dV}, \quad (97.) \qquad y_r = \frac{\int ydV}{\int dV}, \quad (98.) \qquad z_r = \frac{\int zdV}{\int dV}. \quad (99.)$$

The manner in which the body is supposed to be divided to form weight-elements is different in different cases, as certain modes of imaginary division oftentimes simplify the problem very greatly. The integrations must of course be taken between *limits*, as indicated by the conditions of the problem to be solved. For the practical application of the preceding principles the student is referred to the standard treatises on analytical mechanics.

156. Theorems of Pappus. An interesting application of the principles which we have demonstrated in the preceding pages is found in the *centrobaryc method* of determining the volumes of solids of revolution, and the contents of surfaces of revolution. The principles of this method were first stated by Pappus of Alexandria (380 B. C.), and are hence called after him. They are also known as the *Principles of Guldinus*, from a mathematician who republished them about 1640 in a work on centres of gravity of lines, surfaces and solids. The first real demonstration of them, however, appeared in 1647 in a work by Cavalieri.[1] They are as follows: —

I. *The volume of the solid generated by the revolution of a surface about an axis in the same plane with it, is equal to the area of the surface multiplied by the circumference described by its centre of gravity.*

II. *The area of the surface generated by the rotation of a line about an axis lying in the same plane with it, is equal to the product of the line into the circumference described by its centre of gravity.*

Let ABC, Fig. 66, be a surface generating the solid, of which $ABC-DEF$ is a part, by its revolution about the axis X. Any element of ABC, as P, generates a solid by its revolution, as indicated by the dotted lines, and the sum of the solids thus formed by the elements, P, P', etc., of which ABC is composed, is evidently equal to the whole solid of revolution. Supposing these elements to be very small, and denoting by r, r', etc., their distances PR, $P'S$ from the axis X, the prismatic solid described by $P = 2\pi r \times P$; that described by $P' = 2\pi r' \times P'$, and so on, for all the elements. Hence the volume of the whole solid $=$ $2\pi r \times P + 2\pi r' \times P' + 2\pi r'' \times P'' \ldots + 2\pi r^n \times P^n = 2\pi(Pr + P'r' + P''r'' \ldots + P^n r^n)$. Now if the distance of the centre of gravity of ABC from x be denoted by r_0, we have from the demonstration of § 152, p. 78, $Pr + P'r' + P''r' \ldots + P^n r^n = r_0(P + P' + P'' + \ldots P^n)$. If each of these equals be multiplied by 2π, we have $2\pi(Pr + P'r' + P''r'' \ldots + P^n r^n) = 2\pi r_0(P + P' + P'' \ldots + P^n)$. But $P + P' + P'' \ldots + P^n =$ volume of ABC, $2\pi r_0$ is the circumference described by its centre of gravity, and the first member of the equation represents the sum of the solids generated by the elements of ABC, that is, the whole solid of revolution.

Theorem II. follows directly from Theorem I., since a line may be considered as a surface of infinitesimal width, in which case the solid of revolution generated becomes a surface of revolution.

If the surface ABC revolves only through an angle a instead of making a complete revolution, the volume of the solid generated

$$= 2\pi r_0 \times \textit{area of surface} \times \frac{a}{360}.$$

---

[1] See *The Theorem of Pappus*, by J. B. Henck; *Mathematical Monthly*, Vol. I., p. 200.

As an illustration of the preceding let us take the case of the *torus* formed by the revolution of a circle about X. Let the area of the circle be πR^2, and r_0 the distance of its centre from X. Then the volume of the torus generated $= 2\pi r_0 \times \pi R^2$.

Again let it be required to find the area of the circle generated by the revolution of its radius R about X, by the centrobaryc method. The length of the revolving line is R, the distance of its centre of gravity from X, $r_0 = \frac{R}{2}$. Hence the surface generated $= 2\pi \frac{R}{2} \times R = \pi R^2$.

Equilibrium of Bodies as affected by Position of Centre of Gravity.

157. Case of Suspended Bodies. Since a body is upheld when its centre of gravity is supported, the nature of the equilibrium of a body is determined by the position of that point relatively to the points of support. We consider first the case of suspended bodies.

That any suspended body may be in equilibrium, its point of support and centre of gravity must lie in the same vertical, because the resultant weight is then applied directly at the point of suspension, and is balanced by its reaction. Three cases may arise: 1, when the centre of gravity is below the point of suspension; 2, when it is above the point of suspension; 3, when the centre of gravity and point of suspension coincide.

158. Stable Equilibrium. If the centre of gravity is below the point of suspension, the body will return to its former position of equilibrium on being removed from it. The equilibrium in this case is therefore *stable*. Thus let MN, Fig. 67, be a body suspended at S. Let G' be the position of the centre of gravity when removed from its position of equilibrium, so that S and G' are not in the same vertical. The total weight of the body acts directly downward through G', and may be represented by $G'A$. This force may be resolved into two others represented by $G'B$, $G'C$, the former of which simply produces pressure on the pivot at S, by the reaction of which it is balanced, while the latter produces a rotary motion about S as an axis. Hence the body can not come to rest until $G'C$ becomes equal to zero, which occurs when G' takes the position G in the same vertical with S.

159. Unstable Equilibrium. *Unstable equilibrium* occurs when the centre of gravity is above the point of suspension. In this case, if the body be moved so that the two are no longer in the same vertical, it will not return to its former position, but will revolve about the point of suspension till the centre of gravity reaches the lowest possible position. This will be seen by an inspection of Fig. 68, in which $G'A$ represents the weight of the body, $G'B$, $G'C$, its components as before.

160. Neutral or Indifferent Equilibrium. A body suspended at its centre of gravity will remain at rest indifferently in any position, the total weight being always applied at the point of suspension, however the body may be moved. This is *neutral equilibrium*.

The various conditions of equilibrium may be illustrated experimentally by suspending a wheel successively on axes passing above, below and through its centre.

161. Experimental Method of finding Position of Centre of Gravity. From what precedes it appears that the equilibrium of any suspended body is stable only when the centre of gravity is at the lowest possible point, so that if any body be suspended freely its centre of gravity will assume this position. We are thus furnished with a method of finding the centre of gravity by experiment. For example, suppose it is required to find this point in the case of a thin sheet of metal *MN*, Fig. 69. If it be suspended from a point *S* the centre of gravity will assume the lowest possible position. Hence if a plumb-line *SB* also be hung from *S* the centre of gravity of the surface of *MN* must be somewhere on the line *ST*. Marking this line on *MN*, let the body be suspended from some other point, as *S*. The centre of gravity of the surface will now be on *SB*, and as it is also on *ST*, it will be at *G*, the intersection of these two lines. The centre of gravity of the plate must therefore be on a perpendicular to the surface passing through *G*, at a depth equal to half the thickness of the plate.

162. Measure of Stability of Suspended Bodies. The stability of any suspended body is measured by the force required to hold it in an inclined position, which is equal to the force with which it tends to reassume its position of stable equilibrium. The *stability is proportional to the weight of the body multiplied by the distance of the centre of gravity below the axis of suspension.* Let *MN*, Fig. 70, be the body under consideration. *G* its centre of gravity, and S the axis of suspension. The total force tending to move it is the moment of its weight, or $W \times SP$. If now we suppose the centre of gravity to be lowered to G', and the weight to be changed to W', the new moment $= W' \times SP'$. Hence calling M and M' the stabilities in the two cases, we have $M : M' : W \times SP : W' \times SP'$, or as $SP : SP' :: SG ; SG'$, $M : M' :: W \times SG : W' \times SG'$. (100.)

163. Equilibrium of Bodies resting on a Horizontal Plane. A body resting upon a horizontal plane is in equilibrium whenever the vertical passing through the centre of gravity falls within the base, as in that case the weight is directly opposed by the reaction of the supporting surface. Thus if *G* be the centre of gravity of *AB*, Fig. 71, the body will stand, as *GM* falls within the base; but if it be raised to G' by the addition of the piece *AC*, the body will be overturned as $G'M'$ falls without the base. The vertical through the centre of gravity is often called the *line of direction*. The truth of this proposition may be verified by suspending a plumb-line from *G* and G'. If the base of the body be curved, the line of direction must evidently pass through the point in which this base touches the plane on which it rests; otherwise the body will roll until that position is assumed.

The base of a body resting upon a point is evidently that point; if the body is supported by two points, the line joining them may be considered as the base; if on three or more points, the base is the polygon included between the lines joining the points of support.

164. Illustrations and Applications. So long as the vertical through G lies within the base the body is secure. Thus the leaning towers of Pisa and Bologna have stood for centuries, because although they deviate from the perpendicular by a considerable amount, the line of direction still falls so far within their bases that no ordinary shock would be sufficient to overthrow them. The height of the tower at Pisa (which is the *campanile* or bell-tower of the cathedral) is in round numbers 87 m., and the diameter of its base is 17 m. It is inclined 4.7 m. from the perpendicular, but the vertical through the centre of gravity falls 3 m. within the base. The leaning tower of Asinelli at Bologna is 100 m. in height, with an inclination of 1 m. The Garisenda tower in the same city is 63 m. high. with an inclination of 3 m.[1]

The principle explained in the first part of § 163 furnishes another experimental method of obtaining the centre of gravity of a body. This may be done by balancing it on its edge, in which case the centre of gravity must lie in the vertical plane passing through that edge. Balancing a body in three different positions determines its centre of gravity, which must be the point of intersection of three vertical planes passed through the edge in three successive positions of the supported body.

The equilibrium of a body bounded by plane surfaces is stable when the body rests upon a side, and unstable when it rests upon an angle. If a body be rolled it assumes positions of stable and unstable equilibrium alternately. The equilibrium of a body resting on a horizontal surface is evidently *indifferent* so far as motion in a horizontal plane is concerned.

165. Body with curved Surface. — Metacenter. The nature of the equilibrium of a body having a curved surface is determined by the relative position of the centre of gravity and a point known as the *metacenter*. Let ACD, Fig. 72, be a solid bounded by a curved surface, and resting on a plane RQ. That it may be in equilibrium the axis AB must be vertical, as only in that position will the line of direction fall within the base. Suppose now that the body be moved slightly to some new position in which AB is not vertical. It will then be acted upon by two forces, the resultant weight acting downward through G, and the reaction of the surface RQ acting upward along the line PM. The point M, in which AB and PM intersect, is called the *metacenter* of the body.

The forces through G and M form a couple, as indicated by the arrows, the tendency of which is to make ACD return to its original position, so long as the centre of gravity is below *M*. If it assumes the position *G′*, above *M*, the tendency of the couple is to overturn the body, and the equilibrium of the body in its original position is unstable. If *M* and *G* coincide, the body remains at rest in any position.

---

[1] Considerable discrepancy exists among these figures, as given by different authorities. The data above are reduced from *Appleton's American Encyclopedia.*

Hence the equilibrium is ***stable***, ***unstable*** or ***indifferent***, according as the centre of gravity is ***below***, ***above*** or ***coincident with*** the metacenter, or, in general, ***the equilibrium of a body with a curved surface is the same as if it were suspended at its metacenter.***

166. Equilibrium of a Body resting on Inclined Plane. Under these circumstances there is equilibrium when the line of direction falls within the base, precisely as in the case of a body resting on a horizontal surface.

167. General Criterion of Equilibrium. The different conditions of equilibrium demonstrated in the preceding paragraphs may all be comprehended in a single proposition, as follows:— ***In whatever manner a body is supported, the equilibrium is stable when on moving it the centre of gravity ascends, unstable when the centre of gravity descends, and indifferent if it neither ascends nor descends.***

This will be seen to be true of suspended bodies by a simple inspection of Figs. 67, 68. In the case of a body resting on a base, as *AB*, Fig. 71, *G* will evidently rise if *AB* is slightly rotated about either edge of its base. If *AB* rested on an angle, as *B*, the equilibrium would be unstable, and *G* would then occupy its highest possible position, so that on disturbing the body *G* would move in a descending curve. A consideration of Fig. 72 will show that when a body rests on a curved base, if the centre of gravity be below *M*, as at *G*, it moves in an ascending curve on disturbing the body, while if it is at *G′* above *M*, it moves in a descending curve, and if at *M* the motion is in a horizontal line.

If the surface of the body possesses a different curvature in different planes, the body may be in stable, unstable, or indifferent equilibrium, according to the portion of its surface on which it rests. Thus an ellipsoid of revolution is in unstable equilibrium with regard to motion in any plane if placed on either extremity of its longer axis, while if placed on its shorter axis its equilibrium is stable with regard to motion in the vertical plane passing through its major axis, and indifferent with regard to any plane at right angles to this.

168. Experimental Illustrations. The preceding principles explain the deportment of the toys so frequently seen, which represent a grotesque human figure mounted on a curved base. The figure persistently resumes its upright position when laid upon its side. This is explained by the form of the curved surface constituting the base, which is so constructed that the centre of gravity shall be in its lowest possible position when the figure is upright. A method sometimes used to cause an egg to stand on one end is also capable of a similar explanation. The yolk of an egg is contained in a sac nearly concentric with the white, and is somewhat denser than the latter. By a vigorous shaking the yolk-sac may be broken, and the heavy liquid then falls to the lower end of the egg when this is set upright, thus bringing the centre of gravity to a point so low that any motion of the egg tends to raise it. Another experiment often shown is that of causing a wooden cylinder to roll up an inclined plane. This is done by weighting a portion of the cylinder with lead, so that the centre of gravity does not coincide with the centre of magnitude. The cylinder can easily be so placed that the rotation produced by the tendency of the centre of gravity to assume the lowest possible position at the same time causes the body to ascend the plane. The slope of the plane must of course be so small that the rise of the centre of gravity caused by the ascent of the plane shall be less than the fall due to the rotation of the cylinder. A similar explanation applies to the case of two cones united at their bases, which can be made to roll up two inclined planes set at a suitable angle. In this case, as in the last, the centre of gravity really falls, since the rise caused by the rolling of the cones up the plane is less than the fall produced by the simultaneous approach of the points of support to their vertices.

169. Practical Illustrations of Equilibrium. All the attitudes and movements of animals are regulated with regard to the position of the centre of gravity of their bodies. In man, when erect the centre of gravity is about in the median line of the body, but its place changes with every change of position. The whole art in feats of posturing consists in keeping the line of direction within the base formed by the feet. If a limb be extended in any direction the centre of gravity moves in the same direction, but if some other portion of the body be thrown toward the opposite side the equilibrium is preserved. When a porter carries a load upon his back he must bend forward, while a nurse carrying a child in her arms leans backward, otherwise the line of direction falls in one case behind the heels, in the other in front of the toes. For a like reason when climbing a steep hill we bend forward, and on descending it we lean backward. The art of walking on stilts depends on the ability to keep the centre of gravity directly over the narrow base formed by them. The same is true of the art of walking or dancing on the tight rope. In this case the performer generally carries a heavy pole, by moving which in any direction he can more readily bring the centre of gravity to the desired position, thus making his feats far easier and more graceful than if unassisted by this aid. When a quadruped progresses it never raises both feet on one side simultaneously, for in that case the centre of gravity would be unsupported, but the fore foot on one side and the hind foot on the other are lifted together in trotting, the fore foot a little in advance in walking, or else, as in galloping, the two fore legs are lifted, and the body projected forward by the spring of the hind legs. The principles of equilibrium explain the different feats of balancing long poles, swords, etc., in an upright position upon the hand. Feats of this kind are made much easier if a horizontal rotation is given to the body which is balanced, because the centre of gravity describes a small circle about a fixed point, so that even if it is not directly over the base the direction in which it tends to fall is constantly changing, causing a preservation of equilibrium. It is thus that a top remains vertical when spinning rapidly, but falls when at rest. The same explanation applies to the common experiment of balancing a plate upon the point of a sword. This is impossible if the plate is at rest, but if it be made to whirl rapidly no difficulty is experienced.

170. Measure of the Stability of Bodies resting on a Plane.—Statical Stability. We may consider the stability of a body under two aspects: 1st, in relation to the force necessary to move it from its position of equilibrium; 2d, in relation to the force required to completely overturn it. The forces exerted in these two cases determine the *statical* and *dynamical stability of a body.*

Statical Stability. This is measured by the force necessary to begin rotation about an angle of the body. Let MN, Fig. 73, be a mass resting on a horizontal plane, and suppose a force F to be applied along the line CF. If any obstacle be placed at N, so that MN cannot slide along the plane, F will tend to rotate MN about N. The moment of F relatively to an axis through N is $F \times DN$. The moment of the weight W of the body tending to prevent motion is $W \times PN$. When motion begins, $F \times DN = W \times PN$. Hence $W \times PN$ measures the statical stability of MN, and this is jointly proportional to the weight of the body and to the distance of the foot, the vertical through its centre of gravity from the axis about which rotation tends to take place. Hence the greater the weight and the broader the base the greater the statical stability of a body, which is, however, independent of the height of the centre of gravity. If the distance PN is different in different directions, the stability of MN varies according to the direction of the force F.

The product $W \times PN$ is called the *moment of stability* of the body MN.

171. Dynamical Stability. The dynamical stability of a body is measured by the moving force expended in overturning it. To accomplish this in the case of any mass MN, Fig. 74, it must be revolved about one of its angles N, until the centre of gravity, which describes the curve GH, reaches its highest point H, after which the action of gravity will complete the overturning. Hence the whole force is expended in raising G through the vertical height HL. The power required to do this must be proportional to the weight W of the body, and also to the height HL, through which this weight is raised; hence it is measured by their product $W \times HL$. If the centre of gravity occupies a more elevated position G', the distance $H'L'$ will be less than HL, hence less power will in that case be required to overturn the body. Therefore the dynamical stability of a body increases with its weight, and diminishes with the elevation of its centre of gravity.

172. Practical Illustrations of Stability. Many illustrations of the two kinds of stability may be cited. A pyramid owes its great firmness to the breadth of its base and the low position of the centre of gravity. Walls are often strengthened with small expenditure of material by building offsets or by *battering* the side toward which the forces tending to overturn them are directed. A high carriage is upset much more readily than a low one, because the centre of gravity is higher, so that a smaller horizontal force overcomes its dynamical stability, and also if the road is sloping a less inclination is required to throw the line of direction outside of the base. For this reason wagons intended for carrying heavy loads are generally made so as to have the greater part of the load lower than the axles of the wheels. Quadrupeds have a much greater stability than man, because of the broader base formed by their feet, hence their young acquire the art of walking far more rapidly than a child, which must first learn the art of balancing itself on its legs. The human figure is most stable when the feet are so placed as to make the base as large as possible. If the heels are in contact this occurs when the angle of the feet is 90°. If the heels are separated by an amount equal to the length of the foot, their angle should be 60°.

CABOT SCIENCE LIBRARY

CANCELLED
JUN - 4 2007

CABOT
JAN 3 1 2007
BOOK DUE

Phys 227.8
Course in elementary physics.
Cabot Science 003434772
3 2044 091 954 115